A COURNOT-BERTRAND MODEL USING VARIOUS PRODUCTS

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Abstract: *Imperfect competition represent a known issue on modern economic analysis. Monopoly case classical induce a worse solution for consumer side but in the oligopoly competition scenario it will be possible to obtain a solution close to perfect competition situation. In this case, two of the most well-known game theory models in imperfect competition are Cournot model, respectively Bertrand model. First one offers an output equilibrium solution, while the second one, advocating for price competition, highlights an equilibrium solution in terms of price. Alongside these two models, in the last period, we can also note an increasing interest for the Cournot-Bertrand mixed scenario. All these three behavioral types can be identified in the real market. This paper aims to analyze a duopoly market, with linear demand and cost functions, as well as product differentiation characteristics, where a Cournot behavior is adopted by one firm and a Bertrand behavior by the other one. Both firms manage to survive on the market and a stable equilibrium will be manifested if there is sufficient product differentiation. In a very low product differentiation / very high product homogeneity scenario instead, the adjustment process proves to be a divergent one, undermining the equilibrium stability. In a homogenous products context, the Cournot-type firm will produce the perfectly competitive output level and the Bertrand-type competitor will leave the market. The selling price will match the marginal cost and duopoly profit will be null. Even with just one firm on the market, the basic threat of a Bertrand – type potential competitor, warrants a very prudential behavior while also further ensuring the perfect competitive outcome level. The paper is looking to also analyse in detail the existence of Nash equilibrium point, its local stability and uniqueness, as well as the product differentiation impact on this equilibrium and players surviving potential on the market. The final part of the paper contains a simulated study case for price, quantity and profit evolution for various values of model parameters.*

Keywords: *Cournot-Bertrand model; product differentiation; oligopoly theory.*

JEL Classification: *C72; D01; D43; L13.*

1. Introduction

The starting point of oligopoly theory is the Cournot classic model (1838), who derived the Nash equilibrium in a static duopoly game with firms producing perfectly homogeneous goods, choosing simultaneously the output level, in market's demand and price full awareness scenario. At an equilibrium scenario, the price level becomes lower than the monopoly case, but higher than marginal cost. Bertrand (1883) analyzed the same game as his predecessor, choosing the price strategy

instead of output one. He has discovered another Nash equilibrium, price matching the marginal cost and the firms profit being zero. These basic models highlights the idea that the choosing of price or output as strategic variable, in an oligopoly with strategic interaction, has a huge impact on the Nash equilibrium
In the last decades, an increasing interest on the static Cournot-Bertrand model has been manifested. As per its name, the model consider one firm competing in output (Cournot type) and another one, competing in price terms (Bertrand type). Singh and Vives (1984) proved under certain conditions of demand and cost, if two firms can choose to compete in output or price the dominant strategy for both is to compete in output rather than price (Cournot case), in substitute products scenario. Lately, Häckner (2000), Zanchettin (2006), Arya et al. (2008), and Tremblay et al. (2009) explain how exeactly technological and institutional conditions as well as demand assymetries can modify firm profits, optimality being reflected in Bertrand or Cournot-Bertrand model.
Considerind the wide range of theoretical possibilities, Kreps and Scheinkman (1983) assert that firms decision to compete in output or price terms is lastly an empirical question. In the real world, all three behavior's type (Cournot, Bertrand, and Cournot-Bertrand) could be found. Hotels set prices, while flower producers set quantities. In the Japanese electronic industry, Sanyo set prices whilst Panasonic set quantities (Sato, 1996). From this point, any future research on the Cournot-Bertrand model will be warranted and could also improve actual understanding of oligopoly markets.
Tremblay and Tremblay (2011) investigate the Cournot-Bertrand model, some interesting ideas emerging from their research: in homogeneous products case, Cournot-type firm produces the perfect competitive level of market output, whilst Bertrand-type firm leaves the market. The mere threat of a Bertrand-type potential competitor, ensures the perfectly competitive outcome, demonstrating the dramatic effect this one can have on the market power.
The next paragraphs will cover the impact of product differentiation degree on a Cournot-Bertrand static equilibrium model, highlighting certain interesting aspects such as stability, market surviving potential and also product differentiation influence on Nash equilibrium theory. The principles of the related mathematic model are also analised.

2.The Model

The used scenario presents two producers, i and j, competing on the same market, establishing simultaneously their own action paths. First one decides to adopt a Cournot behaviour, competing in quantity terms,whilst the second adopt a Bertrand behaviour, competing in terms of price. The common aim is to maximize their own satisfaction / profit, whilst fully aware of market conditions.
The products may differ by few features and producers can have a high appetite for variety, as per Beath and Katsoulacos (1991). For example, we can mention the real scenario of consumer dilemma, where ask to choose to spend his money in a fast – food or in a coffee shop. Those one may differ in geographic location, quality of

services, atmosphere and despite the fact that consumers will always prefer a certain type of service over the other one, he decide to try both services in a certain time period.
We consider the substitutes products scenario, where above mentioned kind of differentiation can be translated into a linear demand system, as per Dixit (1979),Singh & Vives (1984), Beath & Katsoulacos (1991), Imperato et all (2004), Tremblay (2011) have already mentioned.
The inverse demand functions are:

$$p_1 = a - q_1 - dq_2$$
$$p_2 = a - q_2 - dq_1 \;\rightarrow\; q_2 = \mathrm{a} - p_2 - \mathrm{d} * q_1$$

where a>0 and d ϵ [0, 1]. If d=1, homogeneous products case can be identified whilst each one acts as monopolists when d=0. More precisely, d is an index, whose value is inversely proportional with differentiation degree (differentiation diminishes at d value increases). It also reflects the nature of the products, positive values being specific for substitute products, negatives values for complements scenario, whilst zero values highlights independent products. Demand function decreases in each product's price, but increases/decreases in competitor's price, in substitute/complement products scenario.
In current model, the demand system in strategic variables, q1 and p2 is:

$$\begin{cases} p_1 = \mathrm{a} - \mathrm{ad} + (1 - d^2)q_1 + dp_2 \\ q_2 == \mathrm{a} - p_2 - \mathrm{d}q_1 \end{cases} \quad \text{(see Apendix A)}$$

Cost of production is considered to be linear and identical for both firms c, also matching marginal cost. Thus, the profit function for firm i is:

$$\pi_i = (p_i - c)q_i\,, (\forall) i = \overline{1,2}\,, c \in (0; a)$$

Marginal profit expressions represent the starting point in the best-reply functions determination

$$r_1 : p_2 = \frac{\mathrm{c+ad-a}}{\mathrm{d}} + \frac{2(1-d^2)q_1}{\mathrm{d}} \qquad r_2 : p_2 = \frac{\mathrm{a+c}}{2} - \frac{\mathrm{d}q_1}{2}$$

and further to Nash equilibrium values revealing :

$$p_1^* = \frac{a(2 - d - 2d^2 + d^3) + c(2 + d - d^2 - d^3)}{4 - 3d^2} \qquad p_2^* = \frac{a(2 - d - d^2) + c(2 + d - 2d^2)}{4 - 3d^2}$$

$$q_1^* = \frac{(a-c)(2-d)}{4-3d^2} \qquad q_2^* = \frac{(a-c)(2-d-d^2)}{4-3d^2}$$

$$\pi_1^* = \frac{(a-c)^2(2-d)^2(1-d^2)}{(4-3d^2)^2} \qquad \pi_2^* = \frac{(a-c)^2(2-d-d^2)^2}{(4-3d^2)^2}$$

We further use Dixit's necessary and sufficient stability condition (1986), in order to test the Nash equilibrium stability: $|\pi_{ii}| > |\pi_{ii}|,\ where\ \pi_{ii} = \frac{\partial^2\pi_i}{\partial p_i^2}\ iar\ \pi_{ij} = \frac{\partial^2\pi_i}{\partial p_i \partial p_j}$, $i,j = \overline{1,2}$

$$\begin{cases} \frac{\partial^2\pi_1}{\partial q_1^2} > \frac{\partial^2\pi_1}{\partial q_1 p_2} \\ \frac{\partial^2\pi_2}{\partial p_2^2} > \frac{\partial^2\pi_2}{\partial p_2 q_1} \end{cases} \Longrightarrow \begin{cases} |-2(1-d^2)| > |d| \rightarrow\ 2(1-d^2) > d\ \rightarrow\ 2d^2 + d - 2 < 0 \\ |-2| > |-d| \end{cases} \Longrightarrow$$

$$\begin{cases} \Delta = 17 \rightarrow d_{1,2} = \frac{-1 \pm \sqrt{17}}{4} so\ inecuation\ solution\ is\ d_{1,2} \in \left(\frac{-1-\sqrt{17}}{4}; \frac{-1+\sqrt{17}}{4}\right) \xrightarrow{d\epsilon(0;1)} d \in [0; \frac{\sqrt{17}-1}{4}) \\ d < 2 \xrightarrow{d\epsilon[0;1)} (A) \end{cases}$$

Conclusion: equilibrium is stable $(\forall)d \in \left[0; \frac{\sqrt{17}-1}{4}\right)$ $(\frac{\sqrt{17}-1}{4} \approx 0{,}78)$. More specific, in a differentiated products scenario, the degree of differentiation must be high enough to assure the equilibrium stability (situation reflected by Figure 1, highlated at intersection area of the two isoprofit curves), otherwise we are moving in an instability area, where the adjustment process does not converge to the equilibrium point (see Figure 2).

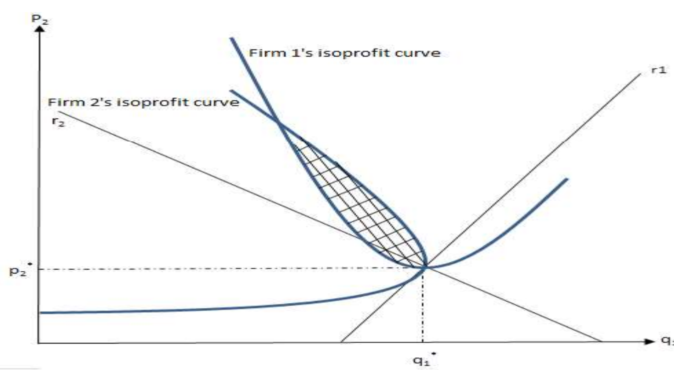

Figure 1: Isoprofit and best – reply functions in the Cournot – Bertrand dupoly
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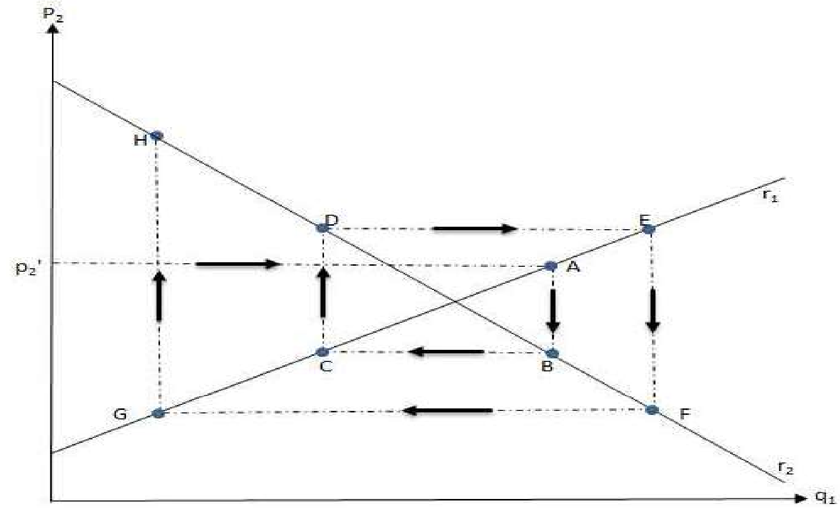

Figure 2: Best – reply functions and equilibrium's instability in the Cournot – Bertrand model
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In Figure 2 , if we start our analyze at a point such as p2′, his competitor's best reply is to produce at the output level that corresponds to point A . In response to firm 1's producing decision, firm 2 sets price at level B. First player's reaction will be to produce the output reflected in point C, the second will react again via D level price and so on. According to the graphic, can be easily obseved that the trend „push" each player's response further away from the best-reply function' intersection point; the equilibrium becomes unstable because the adjustment process prouve to be a divergent one (i.e., it moves from A to B, to C, to D, etc.).
We will further analyze the perfectly substitutes products case (d=1). Whereas $p_1^* = c, p_2^* = c$, the selling price will be identical for both products, also matching marginal cost. We obtain $q_1^* = \mathrm{a} - \mathrm{c},\ q_2^* = 0$, so firm 1 produces the perfectly competitive level of output, whilst firm 2 leaves the market (produces zero output). However, we can highlight that the perfectly competitive market output level is identical in Cournot-Bertrand and Bertrand models ($q^* = \mathrm{a} - c$).Thus, the profit becomes $\pi_1^* = 0, \pi_2^* = 0$, so notwithstanding the output produced by both players, they win nothing in current scenario.

Nash equilibrium can be mathematically expressed by player i scenario of profit maximization, regardless player j behavior, as per below below:

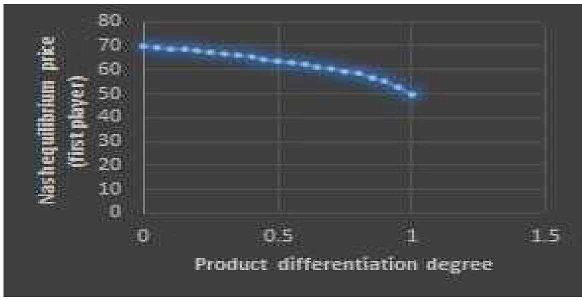
$$\begin{cases} \pi^i\left(q_i^*, p_j^*\right) \geq \pi^i\left(q_i, p_j^*\right) (\forall)\ i, j = \overline{1,2} \\ \pi^j\left(q_i^*, p_j^*\right) \geq \pi^j\left(q_i^*, p_j\right)\ (\forall)\ i, j = \overline{1,2} \end{cases}$$

Proposition: $p_1^* = p_2^* = c$ and $q_1^* = a - c, q_2^*$ =0 defines the only Nash equilibrium.

Proof: we consider the general demand system $p_1 = p_1(q_1, p_2), q_2 = q_2(q_1, p_2)$. Demand functions are differentiable, each having negative slopes ($\frac{\Delta p_1}{\Delta q_1} < 0, \frac{\Delta p_2}{\Delta q_2} < 0$), substitutes products scenario being considered ($\frac{\Delta p_1}{\Delta p_2} > 0, \frac{\Delta q_2}{\Delta q_1} < 0$). If Q_{pc} = perfectly competitive output level, we further analyze the following possible situations:

- $q_1 > Q_{\mathrm{pc}}$. Considering the negatively sloped demand function, current output level leads to negative profits for first player (p(q_1 > Qpc) < c). A better alternative would be leaving the market and earn zero profit.
- $q_1 < Q_{\mathrm{pc}}$. Zero output level produced by second player, also negative slope demand function leads to p(q_1 < Qpc) > c. This time, second player best response , would be p_2 = p(q_1 < Qpc) – ε > c(ε > 0), actual scenario offering the possibility of earning positive profit (q_2 > 0) for both firms. On the other side, $p_2 > c$ offer firm's 1 the chance of increasing its production and win extra profit,by supplying the entire demand at p_2. No residual demand will remain for firm 2, a strong enough incentive to further decrease its price; raising q_1 and lowering p_2 actions will continue until q_1 = Q_{pc} and p_2 = c;
- $p_2 < c$. Firm 2 earns negative profit, then will decide to leave the market and obtain zero profit;
- $p_2 > c$. As we have mentioned previously, firm 1's best reply to p_2 > c is to produce all market's requested output at p_2, such that q_1 < Q_{pc}. In return, will became profitable for firm 2 to decrease price level. The process of output increasing and price reducing will continue until q_1 = Q_{pc} and p_2 = c.
- $q_1 = Q_{\mathrm{pc}}\ and\ p_2 = c$. Was already demonstrated that first firm cannot increase its profit in $q_1 \neq Q_{\mathrm{pc}}$ case, and neither second one, if $p_2 \neq c$. Any action path players would choose, would lead to not a higher profit level then the one expected from its current strategy, therefore they are not stimulated to modify the quantity/price triggering the unique Nash equilibrium point.

All previous analysis are meant to highlight the dramatic effect that a potential competitor can induce in a market. The mere threat of a price competitor offering a homogeneous product, ensures that a monopolist will adopt a perfectly competitive firm's behavior. Briefly, the potential competitor fully annihilate market power.
We further analyze, via graphical representation, the price/quantity/profit sensitivity to the changes in the product differentiation levels (d parameter values) in a Nash equilibrium scenario. Starting from Appendix B tabled values and also customizing parameters a and c (a=90 EUR, c=50 EUR) we gradually increase product homogeneity degree by ratio of 0.05 (from independent products case (d = 0) to homogeneous products scenario (d = 1))



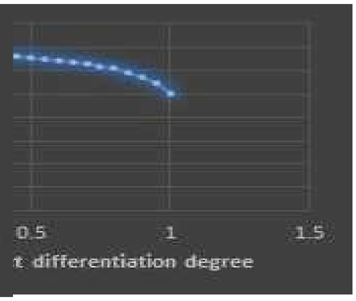


Figure 3:Nash equilibrium price evolution (player i)
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Figure 4:Nash equilibrium price evolution (player j)
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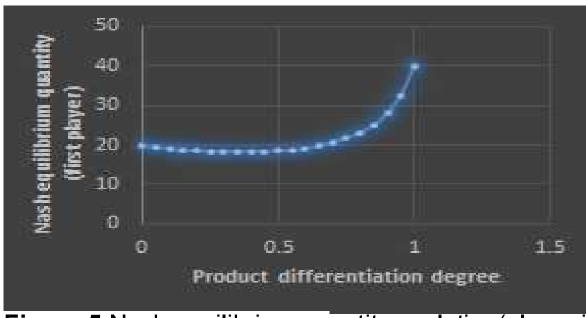

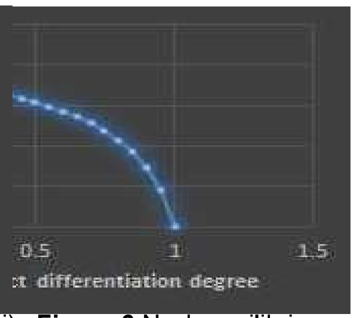

Figure 5:Nash equilibrium quantity evolution(player i)
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Figure 6:Nash equilibrium quantity evolution(player j)
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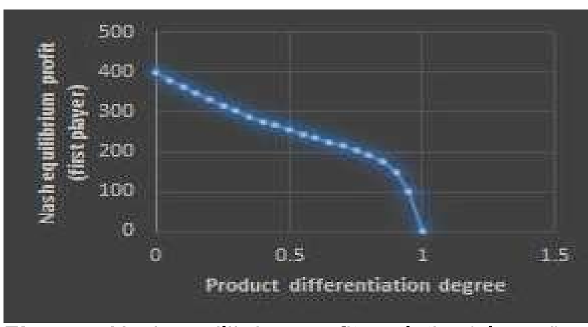

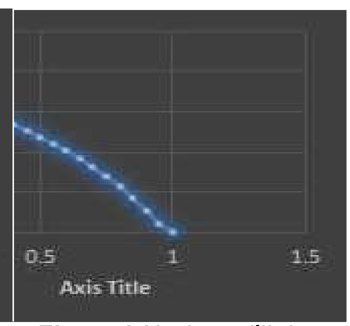

Figure 7:Nash equilibrium profit evolution(player i)
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Figure 8:Nash equilibrium profit evolution(player j)
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3. Conclusions

In independent products scenario (d = 0), a and c coefficients are identical, following different trendlines as the level of products differentiation decreases, although their sum remains unitary, as $\frac{2-d-2d^2+d^3}{4-3d^2}+\frac{2+\mathrm{d}-d^2-d^3}{4-3d^2}=\frac{2-d-d^2}{4-3d^2}+\frac{2+d-2d^2}{4-3d^2}=1$. As a>c, we assist at the gradual price decrease, from a and c average value of 70 EUR, down to marginal cost level of 50 EUR;

As for the quantities triggering the equilibrium scenario, different behaviours can be observed in (0;1) interval. First player level of output will decrease slowly from its 20 EUR initial value (tangible in independent good case), down to 18,16 EUR, as long as the product differentiation degree is not higher than $\frac{6-2\sqrt{6}}{3}\approx 0{,}37$; once d value passes through $\left[\frac{6-2\sqrt{6}}{3};1\right]$ area, the trend will become ascending, triggering the perfectly competitive 40 EUR output level in the absence of product differentiation. The explanation is mathematical as well (Appendix C), highlighting the fact that for $q^{*\prime}=-\frac{(a-c)(3d^2-12\mathrm{d}+4)}{(4-3d^2)^2}$ the unique critical point (minimum point as well) beeing previously mentioned. Second player will gradually decrease the output level down to zero value, once the product homogenity level starts to increase, the downward trend mentaining in all [0;1] area.

Profits for equilibrium scenario follow a downward trend each, from 0.25 (a-c)^2 down to zero value in homogeneous products case. One more time, math principles offer the key of this behaviour's understanding, as ${\pi_1}^{*\prime}=\frac{-2(\mathrm{a-c})^2(2-d)(4-3d^2)(6d^3-5d^2-4d+4)}{(4-3d^2)^2}, {\pi_2}^{*\prime}=\frac{-2(\mathrm{a-c})^2(\mathrm{d}+2)(1-\mathrm{d})(4-3d^2)(3d^2-4d+4)}{(4-3d^2)^2}$, strictly negative expression reflecting decreasing functions (Appendix D). Furthermore the graphical analyse highlights the decreasing trend of profits, from 400 EUR down to the breakeven point (zero profit).

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Appendix A

$p_1 = a - q_1 - dq_2$ ➔ $p_1 = a - q_1 - d(a - p_2 - dq_1) =$

$a - q_1 - ad + dp_2 + d^2 q_1$

$p_2 = a - q_2 - dq_1 \rightarrow q_2 = \mathrm{a} - p_2 - \mathrm{d}q_1$

$p_1 = a - ad + dp_2 - (1 - d^2)q_1$

$\pi_1 = (p_1 - c)q_1 = [\mathrm{a} - \mathrm{ad} + dp_2 - (1 - d^2)q_1 - c]q_1 = aq_1 - adq_1 - q_1^2 + d^2 q_1^2 + dp_2 q_1 - cq_1 = q_1^2(d^2 - 1) + q_1(a - ad + dp_2 - c)$

$\pi_2 = (p_2 - c)q_2 = \mathrm{a}p_2 - p_2^2 - dp_2 q_1 - ac + cp_2 + cdq_1 = -p_2^2 + p_2(a - dq_1 + c) - ac + cdq_1$

$$\begin{cases} \frac{\partial \pi_1}{\partial q_1} = 2q_1(d^2 - 1) + \mathrm{a} - \mathrm{ad} + \mathrm{d}p_2 - \mathrm{c} = 0 \\ \frac{\partial \pi_2}{\partial p_2} = -2p_2 + \mathrm{a} - \mathrm{d}q_1 + \mathrm{c} = 0 \end{cases} ➔ \begin{cases} p_2 = \frac{\mathrm{c+ad-a}}{\mathrm{d}} + \frac{2(1-d^2)q_1}{\mathrm{d}} \\ p_2 = \frac{\mathrm{a+c}}{2} - \frac{\mathrm{d}q_1}{2} \end{cases}$$

By substitution:

$$\frac{\mathrm{c + ad - a}}{\mathrm{d}} + \frac{2(1 - d^2)q_1}{\mathrm{d}} = \frac{\mathrm{a + c}}{2} - \frac{\mathrm{d}q_1}{2} \rightarrow 2c + 2ad - sa + 4(1 - d^2)q_1 = ad + cd - d^2 q_1$$

$$(4 - 3d^2)q_1 = -ad + cd = 2a - 2c \rightarrow q_1^* = \frac{-d(\mathrm{a - c}) + 2(a - c)}{4 - 3d^2} = \frac{(\mathrm{a - c})(2 - \mathrm{d})}{4 - 3d^2}$$

$$p_2 = \frac{\mathrm{a+c}}{2} - \frac{\mathrm{d}}{2}\frac{(\mathrm{a-c})(2-\mathrm{d})}{4-3d^2} = \frac{4a-3ad^2+4c-3cd^2-2a+ad^2+2cd-cd^2}{2(4-3d^2)} = \frac{2(2\mathrm{a}+2c-ad^2-2cd^2-ad+cd)}{2(4-3d^2)} = \frac{2(2\mathrm{a}+2c-ad^2-2cd^2-ad+cd)}{2(4-3d^2)} \rightarrow p_2^* = \frac{a(2-d-d^2)+c(2+d-2d^2)}{4-3d^2}$$

$$q_2 = \mathrm{a} - p_2^* - \mathrm{d}q_1^* = \frac{a(4 - 3d^2) - \mathrm{a}(2 - \mathrm{d} - d^2) - c(2 + d - 2d^2) - d(2a - ad - 2c + cd)}{4 - 3d^2}$$

$(4 - 3d^2)\, q_2 = 4a - 3ad^2 - 2a + ad + ad^2 - 2c - cd + 2cd^2 - 2ad + ad^2 + 2cd - cd^2$

$$(4-3d^2)\, q_2 = 2a - 2c - ad^2 + cd^2 - ad + cd = 2(a-c) - d(a-c) - d(a-c)$$

$$q_2^* = \frac{(\mathrm{a}-\mathrm{c})(2-\mathrm{d}-d^2)}{4-3d^2}$$

$$p_1 = \mathrm{a} - \mathrm{ad} - q_1^*(1-d^2) + \mathrm{d}p_2^*$$
$$= \mathrm{a} - \mathrm{ad} - \frac{(\mathrm{a}-\mathrm{c})(2-\mathrm{d})}{4-3d^2}(1-d^2) + d\frac{\mathrm{a}(2-\mathrm{d}-d^2) + c(2+d-2d^2)}{4-3d^2}$$

$$(4-3d^2)\, p_1 = 4a - 3ad^2 - 4ad + 3ad^3 - 2a + ad + 2c - cd + 2ad^2 - ad^3 - 2cd^2 + cd^3 + 2ad - ad^2 - ad^3 + 2cd + cd^2 - 2cd^3$$
$$= 2a + 2c - 2ad^2 - ad + ad^3 + cd - cd^2 - cd^3$$

$$(4-3d^2)\, p_1 = a(2-d-2d^2+d^3) + c(2+d-d^2-d^3)$$

$$p_1^* = \frac{a(2-d-2d^2+d^3) + c(2+d-d^2-d^3)}{4-3d^2}$$

$$\pi_1 = (p_1^* - c)q_1^* = \frac{a(2-d-2d^2+d^3) + c(2+d-d^2-d^3)}{4-3d^2}\,\frac{(\mathrm{a}-\mathrm{c})(2-\mathrm{d})}{4-3d^2}$$

$$\pi_1 = \frac{(\mathrm{a}-\mathrm{c})^2(d^3-2d^2-d+2)(2-d)}{(4-3d^2)^2} = \frac{(\mathrm{a}-\mathrm{c})^2[d^2(\mathrm{d}-2)-(\mathrm{d}-2)](2-\mathrm{d})}{(4-3d^2)^2}$$

$$\pi_1^* = \frac{(\mathrm{a}-\mathrm{c})^2(2-d)^2(1-d^2)}{(4-3d^2)^2}$$

$$\pi_2 = (p_2^* - c)q_2^* = \frac{a(2-d-d^2) + c(d^2+d-2)}{4-3d^2}\,\frac{(\mathrm{a}-\mathrm{c})(2-\mathrm{d}-d^2)}{4-3d^2}$$

$$\pi_2 = \frac{(\mathrm{a}-\mathrm{c})^2(-d^2-d+2)(2-d-d^2)}{(4-3d^2)^2}$$

$$\pi_2^* = \frac{(\mathrm{a}-\mathrm{c})^2(2-\mathrm{d}-d^2)^2}{(4-3d^2)^2}$$

If d=0 then $p_1^* = p_2^* = \frac{a+c}{2}$, $q_1^* = q_2^* = \frac{a-c}{2}$ and $\pi_1^* = \pi_2^* = \frac{(\mathrm{a}-\mathrm{c})^2}{4}$
If d=1 then $p_1^* = p_2^* = c$, $q_1^* = a-c,\ q_2^* = 0$ and $\pi_1^* = \pi_2^* = 0$.

Appendix B

Table 1: Simulation of price, quantity and profit evolution

d	p_1	p_2
0	0.5*a+0.5*c	0.5*a+0.5*c
0.05	0.487195*a+0.512805c	0.48779*a+0.51221c
0.1	0.473804*a+0.526196*c	0.476071*a+0.523929*c
0.15	0.459854*a+0.540146*c	0.464717*a+0.535283*c
0.2	0.445361*a+0.554639*c	0.453608*a+0.546392*c
0.25	0.430328*a+0.569672*c	0.442623*a+0.557377*c
0.3	0.414745*a+0.585255*c	0.431635*a+0.568365*c
0.35	0.398589*a+0.601411*c	0.420509*a+0.579491*c
0.4	0.381818*a+0.618182*c	0.409091*a+0.590909*c
0.45	0.36437*a+0.63563*c	0.3972*a+0.6028*c
0.5	0.346154*a+0.653846*c	0.384615*a+0.615385*c
0.55	0.327041*a+0.672959*c	0.371059*a+0.628941*c
0.6	0.306849*a+0.693151*c	0.356164*a+0.643836*c
0.65	0.285316*a+0.714684*c	0.339433*a+0.660567*c
0.7	0.262055*a+0.737945*c	0.320158*a+0.679842*c
0.75	0.236486*a+0.763514*c	0.297297*a+0.702703*c
0.8	0.207692*a+0.792308*c	0.269231*a+0.730769*c
0.85	0.174147*a+0.825853*c	0.233288*a+0.766712*c
0.9	0.133121*a+0.866879*c	0.184713*a+0.815287*c
0.95	0.079207*a+0.920793*c	0.11412*a+0.88588*c
1	c	c

d	q_1	q_2	π_1	π_2
0	0.5*(a-c)	0.5*(a-c)	0.25*$(a-c)^2$	0.25*$(a-c)^2$
0.05	0.488416*(a-c)	0.48779*(a-c)	0.237954*$(a-c)^2$	0.237939*$(a-c)^2$
0.1	0.478589*(a-c)	0.476071*(a-c)	0.226757*$(a-c)^2$	0.226643*$(a-c)^2$
0.15	0.470439*(a-c)	0.464717*(a-c)	0.216333*$(a-c)^2$	0.215962*$(a-c)^2$
0.2	0.463918*(a-c)	0.453608*(a-c)	0.206611*$(a-c)^2$	0.20576*$(a-c)^2$
0.25	0.459016*(a-c)	0.442623*(a-c)	0.197528*$(a-c)^2$	0.195915*$(a-c)^2$
0.3	0.455764*(a-c)	0.431635*(a-c)	0.189026*$(a-c)^2$	0.186309*$(a-c)^2$
0.35	0.454233*(a-c)	0.420509*(a-c)	0.181052*$(a-c)^2$	0.176828*$(a-c)^2$
0.4	0.454545*(a-c)	0.409091*(a-c)	0.173554*$(a-c)^2$	0.167355*$(a-c)^2$
0.45	0.45689*(a-c)	0.3972*(a-c)	0.166477*$(a-c)^2$	0.157768*$(a-c)^2$
0.5	0.461538*(a-c)	0.384615*(a-c)	0.159763*$(a-c)^2$	0.147929*$(a-c)^2$
0.55	0.468876*(a-c)	0.371059*(a-c)	0.153342*$(a-c)^2$	0.137685*$(a-c)^2$
0.6	0.479452*(a-c)	0.356164*(a-c)	0.14712*$(a-c)^2$	0.126853*$(a-c)^2$
0.65	0.494053*(a-c)	0.339433*(a-c)	0.140961*$(a-c)^2$	0.115215*$(a-c)^2$
0.7	0.513834*(a-c)	0.320158*(a-c)	0.134653*$(a-c)^2$	0.102501*$(a-c)^2$
0.75	0.540541*(a-c)	0.297297*(a-c)	0.127831*$(a-c)^2$	0.088386 $(a-c)^2$
0.8	0.576923*(a-c)	0.269231*(a-c)	0.119822*$(a-c)^2$	0.072485*$(a-c)^2$
0.85	0.627558*(a-c)	0.233288*(a-c)	0.109288*$(a-c)^2$	0.054435*$(a-c)^2$
0.9	0.700637*(a-c)	0.184713*(a-c)	0.09327*$(a-c)^2$	0.034119*$(a-c)^2$
0.95	0.812379*(a-c)	0.11412*(a-c)	0.064346*$(a-c)^2$	0.013023*$(a-c)^2$
1	a-c	0	0	0

Source: own processing

Appendix C

$$q_1^* = \frac{(a-c)(2-d)}{4-3d^2} \rightarrow {q_1}^{*\prime} = \frac{\Delta q_1^*}{\Delta d} = (a-c)\frac{-(4-3d^2)-(2-d)(-6d)}{(4-3d^2)^2}$$
$$= (a-c)\frac{-4+3d^2+12d-6d^2}{(4-3d^2)^2} = -\frac{(a-c)(3d^2-12\text{d}+4)}{(4-3d^2)^2}$$

Ecuation $3d^2 - 12\text{d} + 4 = 0$ has $\Delta = 96 \rightarrow d_{1,2} = \frac{6\pm2\sqrt{6}}{3} then$

$$\xrightarrow{d\epsilon(0;1)} \begin{cases} 3d^2 - 12\text{d} + 4 > 0\ (\forall)\ d \in [0; \frac{6-2\sqrt{6}}{3}) \\ 3d^2 - 12\text{d} + 4 \le 0\ (\forall)\ d \in [\frac{6-2\sqrt{6}}{3}; 1] \end{cases} \rightarrow \begin{cases} {q_1}^{*\prime} < 0\ (\forall) d \in [0; \frac{6-2\sqrt{6}}{3}) \\ {q_1}^{*\prime} \ge 0\ (\forall) d \in [\frac{6-2\sqrt{6}}{3}; 1] \end{cases} \rightarrow$$
$$\begin{cases} q_1^* \downarrow\ (\forall) d \in [0; \frac{6-2\sqrt{6}}{3}) \\ q_1^* \uparrow\ (\forall) d \in [\frac{6-2\sqrt{6}}{3}; 1] \end{cases}$$

$$q_2^* = \frac{(\text{a}-\text{c})(2-\text{d}-d^2)}{4-3d^2} \rightarrow {q_2}^{*\prime} = \frac{\Delta q_1^*}{\Delta d}$$
$$= (\text{a}-\text{c})\frac{-(1-2d)(4-3d^2)-(2-d-d^2)(-6d)}{(4-3d^2)^2}$$
$$= (a-c)\frac{-4+3d^2-8d+6d^3+12d-6d^2-6d^3}{(4-3d^2)^2}$$
$$= -\frac{(a-c)(3d^2-4\text{d}+4)}{(4-3d^2)^2}$$

Ecuation $3d^2 - 4\text{d} + 4 = 0$ has $\Delta < 0 \rightarrow 3d^2 - 4\text{d} + 4 > 0 (\forall)\ d \in [0;1]\ then\ \ {q_2}^{*\prime} < 0\ (\forall) d \in [0;1]\ \rightarrow q_2^* \downarrow\ (\forall)\ d \in [0;1]$.

Appendix D

$$\pi_1^* = \frac{(\mathrm{a}-\mathrm{c})^2(2-d)^2(1-d^2)}{(4-3d^2)^2} \to {\pi_1}^{*\prime} = \frac{\Delta\pi_1^*}{\Delta d}$$

$$= (\mathrm{a}-\mathrm{c})^2 \frac{[-2(2-d)(1-d^2)+(2-\mathrm{d})^2(-2d)](4-3d^2)^2 - 2(2-\mathrm{d})^2(1-d^2)(4-3d^2)(-6d)}{(4-3d^2)^2}$$

$$= (\mathrm{a}-\mathrm{c})^2 \frac{-2(2-d)[(1-d^2)+2\mathrm{d}-d^2](4-3d^2)^2 - 2(2-\mathrm{d})^2(1-d^2)(4-3d^2)(-6d)}{(4-3d^2)^2}$$

$$= (\mathrm{a}-\mathrm{c})^2 \frac{-2(2-d)(4-3d^2)[(1+2d-2d^2)(4-3d^2)+(2-\mathrm{d})(1-d^2)(-6d)}{(4-3d^2)^2}$$

$$= (\mathrm{a}-\mathrm{c})^2 \frac{-2(2-d)(4-3d^2)(4+8d-8d^2-3d^2-6d^3+6d^4-12d+12d^3+6d^2-6d^4)}{(4-3d^2)^2}$$

$$= (\mathrm{a}-\mathrm{c})^2 \frac{-2(2-d)(4-3d^2)(6d^3-5d^2-4d+4)}{(4-3d^2)^2}$$

If we derivate term $6d^3-5d^2-4d+4$, we obtain $18d^2-10d-4 \to \Delta = 388 \to d_{1,2} = \frac{5\pm\sqrt{97}}{18}$

$$\xRightarrow{d\epsilon(0;1)} \begin{cases} 18d^2-10d-4 < 0 \ (\forall)\ d \in [0; \frac{5+\sqrt{97}}{18}) \\ 18d^2-10d-4 \geq 0 \ (\forall)\ d \in [\frac{5+\sqrt{97}}{18}; 1] \end{cases}$$

$$\to \begin{cases} 6d^3-5d^2-4d+4 \downarrow 0 \ (\forall) d \in [0; \frac{5+\sqrt{97}}{18}) \\ 6d^3-5d^2-4d+4 \uparrow 0 \ (\forall) d \in [\frac{5+\sqrt{97}}{18}; 1] \end{cases}$$

If we note f(d) = $6d^3-5d^2-4d+4$, we have f(0)>0, f($\frac{5+\sqrt{97}}{18}$)>0, f(1)>0, meaning that $6d^3-5d^2-4d+4 > 0 \ (\forall) d \in [0;1] \to {\pi_1}^{*\prime} < 0 \ (\forall) d \in [0;1] \to {\pi_1}^{*} \downarrow \ (\forall) d \in [0;1]$

$$\pi_2^* = \frac{(\mathrm{a}-\mathrm{c})^2(2-\mathrm{d}-d^2)^2}{(4-3d^2)^2} \rightarrow {\pi_2}^{*\prime} = \frac{\Delta\pi_2^*}{\Delta d}$$
$$= (\mathrm{a}-\mathrm{c})^2 \frac{2(2-\mathrm{d}-d^2)(-1-2d)(4-3d^2)^2 - 2(2-\mathrm{d}-d^2)^2(4-3d^2)(-6d)}{(4-3d^2)^2}$$
$$= (\mathrm{a}-\mathrm{c})^2 \frac{2(2-\mathrm{d}-d^2)(4-3d^2)(-4+3d^2-8d+6d^3+12d-6d^2-6d^3}{(4-3d^2)^2}$$
$$= -(\mathrm{a}-\mathrm{c})^2 \frac{2(\mathrm{d}+2)(1-\mathrm{d})(4-3d^2)(3d^2-4d+4)}{(4-3d^2)^2}$$

Equation $3d^2 - 4\mathrm{d} + 4 = 0$ has $\Delta < 0 \rightarrow 3d^2 - 4\mathrm{d} + 4 > 0 (\forall)\ d \in [0;1]\ then\ \ {\pi_2}^{*\prime} < 0\ (\forall) d \in [0;1]\ \rightarrow \pi_2^* \downarrow\ (\forall)\ d \in [0;1]$.