TRADITIONAL VS. FUZZY INDICATORS OF MODERN PORTFOLIO THEORY

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Abstract: *This paper offers another perspective upon the well-known indicators of Modern Portfolio Theory (created by Harry Markowitz): arithmetic mean or geometric mean for return on financial assets, standard deviation or variance for financial risk, and covariance or correlation between the assets included in the portfolio. This perspective consists of modelling these statistical indicators, using the triangular fuzzy numbers, due to the advantages they have. The first advantage of the fuzzy approach is the returns on financial assets, and the financial asset risks are expressed in intervals with minimum and maximum values, called the triangular fuzzy numbers. This advantage makes the decision of investment more accurate, especially considering the volatility of financial assets. Using triangular fuzzy numbers in estimating the returns based on history of trading, can overcame the fact that past performance is no guarantee for future results, due to the different possibilities of fuzzification. In this paper, the returns will be fuzzified considering the mode value (the most frequent value in a given period of time) of the returns from the past period. This statistical value will help the investors to evaluate the frequency of the last returns and to estimate the most probably frequent value of the returns for the next period. So, the estimation, not only the decision, will be more reliable. The second advantage of using the triangular fuzzy numbers in modelling the financial return and the financial risk is their membership function, which allows the investors to evaluate their investments, depending on the membership degree. The returns of the assets that are closer to the mode return, will be most likely the returns for the next period. The returns that are closer to the limits of the fuzzy intervals, will be less probably the returns for the next period. This assumption has its own gap: the market should have the same conditions as the last period.*

Keywords: *triangular fuzzy number (TFN); performance indicators; financial return; financial risk; financial assets.*

JEL Classification: *G11; G10.*

1. Introduction

The portfolio selection theory comes from the assumption that investors seek to maximize the return while minimizing the risk of investments, and they are willing to assume a higher risk only if a higher expected return compensates the risk. Based on this assumption, the investor`s portfolio can be analyzed within two balanced dimensions: the expected return of the portfolio and the risk of the portfolio. The expected return is calculated as a weighted mean of the expected individual returns. The financial risk - synonymous with volatility in Markowitz`s theory - is measured by these tools: ” (1) calculation of expected return, (2) the variance of an expected return; (3) the standard deviation from an expected return, (4) the covariance of a portfolio of securities, and (5) the correlation between investments ” (Mangram, 2013).

2. Literature review

The portfolio diversification was a subject of interest for many researchers. Markowitz was the author of the portfolio diversification theory or Modern Portfolio Theory (MPT), which was developed by many following researchers. Sharpe (1964) built CAPM (Capital Assets Price Model), Samuelson and Merton (1969) used the stochastic programming in diversification, Watada (1997) proposed the vague goals of the financial return and risk in portfolio selection. Carlsson et al. (2002) introduced the trapezoidal fuzzy number as financial returns in portfolio selection; Huang (2006) determined financial return assets as stochastic variables with fuzzy information and Liagkouras and Metaxiotis (2018) proposed the use of fuzzy logic for mean-variance portfolio optimization with transaction costs. (Sun et al., 2019:6).

Using fuzzy logic in Modern Portfolio Theory is a challenging direction for portfolio optimization, due to the capacity of fuzzy logic to manage the vagueness of input parameters (expected return and financial risk), in order to minimize the estimation errors or to avoid the phenomenon of “Markowitz optimization enigma,” coined by Michaud in 1989 (Marakbi, 2016).

3. Modelling Triangular Fuzzy Financial Asset Return and Risk

Modeling financial assets and risks with triangular fuzzy numbers is due to the volatility of financial assets that form a portfolio. This volatility of assets determines different levels of the intensities of financial return and risk: high, low, or uncertain values of these intensities. In order to analyze the volatility of assets, the first step is to transform real numbers in fuzzy numbers and to create an adequate membership function. Thus, financial return and financial risk have to be transformed from the Markowitz form to fuzzy form.

Definition 1: Let the financial asset return on the financial market be R_a, and let F [0,1] be the rules set for all fuzzy triangular numbers. The fuzzy number (R_a) is considered the triangular number of the financial asset return: $R_a = \{r, \mu_r / r \in R_a\}$, where $\mu_r : R_a \to [0,1]$, if the membership function is the following:

$$\mu_r = \begin{cases} 1 - \dfrac{r_x - r_a}{r_b - r_a}, for \quad r_a \le r_x \le r_b \\ 1, for \quad r_x = r_b \\ 1 - \dfrac{r_c - r_x}{r_c - r_b}, for \quad r_b \le r_x \le r_c \\ 0, for\ the\ values\ out\ of\ the\ range\ [0,1] \end{cases}$$

Definition 2: Let the triangular fuzzy number that defines the financial asset return be of the form: $R_a = \{r, \mu_r / r \in R_a\}$, for every $i = \overline{1, n}$. The set $[R_a]^\alpha = [R_{a1}(\alpha), R_{a2}(\alpha)]$, for every $i = \overline{1, n}$ is considered the level set of triangular fuzzy number R_a, where:

$$R_{a1}(\alpha) = (r_b - r_a)\alpha + r_a;$$
$$R_{a2}(\alpha) = r_c - (r_c - r_b)\alpha.$$

Definition 3: Let the financial asset risk on the financial market be (σ_a), and let F [0,1] be the rules set for all fuzzy triangular numbers. The fuzzy number (σ_a) is considered the triangular number of the financial asset risk: $\sigma_a = \{\sigma, \mu_\sigma / \sigma \in \sigma_a\}$, where $\mu_\sigma : \sigma_a \to [0,1]$ if the membership function is the following:

$$\mu_\sigma = \begin{cases} 1 - \dfrac{\sigma_x - \sigma_a}{\sigma_b - \sigma_a}, for \quad \sigma_a \le \sigma_x \le \sigma_b \\ 1, for \quad \sigma_x = \sigma_b \\ 1 - \dfrac{\sigma_c - \sigma_x}{\sigma_c - \sigma_b}, for \quad \sigma_b \le \sigma_x \le \sigma_c \\ 0, for\ the\ values\ out\ of\ the\ range\ [0,1] \end{cases}$$

Definition 4: Let the triangular fuzzy number that defines the financial asset return be of the form: $\sigma_a = \{\sigma, \mu_\sigma / \sigma \in \sigma_a\}$, for every $i = \overline{1, n}$. The set $[\sigma_a]^\alpha = [\sigma_{a1}(\alpha), \sigma_{a2}(\alpha)]$, for every $i = \overline{1, n}$ is considered the level set of triangular fuzzy number σ_A, where:

$$\sigma_{a1}(\alpha) = (\sigma_b - \sigma_a)\alpha + \sigma_a;$$
$$\sigma_{a2}(\alpha) = \sigma_c - (\sigma_c - \sigma_b)\alpha.$$

Observation 1: Financial asset return may take values in the range $[r_x - r \le r \le r + r_x]$ and financial asset risk may take values in the range $[\sigma_x - \sigma \le \sigma \le \sigma + \sigma_x]$, where $r_x, \sigma_x \in R$.

The two triangular fuzzy numbers that characterize the return on asset and the financial risk are as follows:

$R_{ai} = (r_{ai} \quad r_{bi} \quad r_{ci})$, for every $i = \overline{1, n}$;

$\sigma_{ai} = (\sigma_{ai} \quad \sigma_{bi} \quad \sigma_{ci})$, for every $i = \overline{1, n}$.

4. Modelling the financial expected return with fuzzy triangular numbers

4.1. Traditional versus triangular fuzzy return on financial asset

The financial asset return is a performance indicator that provides information about the earnings that investors gained over a period. Assuming that no dividends were paid over the period, the formula of financial asset return is $R_a = (P_{t1} - P_{t0}) / P_{t0}$ (Chen, 2013), where P_t and P_{t-1} is the price of the asset in time *t* and *t-1*. If dividends were paid, the formula of financial asset return is $R_a = (P_t - P_{t0} + D_1) / P_{t0}$, where D_1 is dividend at time *t*. The return of a financial asset over a time horizon *t* (R_{ai}) can be described as follows:

$$R_{ai} = \begin{pmatrix} t_0 & t_1 & t_3 & \dots & t_n \\ R_{a1} & R_{a2} & R_{a3} & \dots & R_{an} \end{pmatrix}$$

Proposition 1: The most frequent value of the return in the last period will be probably the most frequent value of the return for the next period. (if the market conditions remain the same).

This proposition can be illustrated by the following example:

Example 1: There are two financial assets A_1 and A_2, which are included in a portfolio. The data about their returns in the year N are as follows:

$$R_{a1} = \begin{pmatrix} M_1 & M_2 & M_3 & M_4 & M_5 & M_6 & M_7 & M_8 & M_9 & M_{10} & M_{11} & M_{12} \\ 0.1 & 0.4 & 0.6 & 0.5 & 0.3 & 0.7 & 0.7 & 0.8 & 0.7 & 0.7 & 0.6 & 0.7 \end{pmatrix}$$

$$R_{a2} = \begin{pmatrix} M_1 & M_2 & M_3 & M_4 & M_5 & M_6 & M_7 & M_8 & M_9 & M_{10} & M_{11} & M_{12} \\ 0.5 & 0.5 & 0.3 & 0.4 & 0.6 & 0.7 & 0.5 & 0.6 & 0.7 & 0.8 & 0.5 & 0.9 \end{pmatrix}$$

From this data series, the most frequent value of the returns for A_1 and A_2 is 0.7 and 0.5. However, there are also maximum and minimum values for the returns of these assets. Considering this volatility, the returns for these assets may be fuzzified in the triangular fuzzy number as: R_{a1} = (0.1 0.7 0.8) and R_{a2} = (0.3 0.5 0.9). The minimum value of the fuzzy number represents the minimum value of assets return in the year N, while the middle value is obtained through the mode function, which is the most frequently occurring return in the year N. The membership function of these financial assets A_1 and A_2 represents the probability degree of return realization. Because of its frequency, the mode value of the assets represents the highest degree of realisation, and the margins of the fuzzy numbers represent the lowest grade of realisation (the probability of obtaining 0,8 for A_1 and 0,9 for A_2 in next period, is very low, because of the frequency of these values). The membership functions of these fuzzy returns can be described in Figure 1.

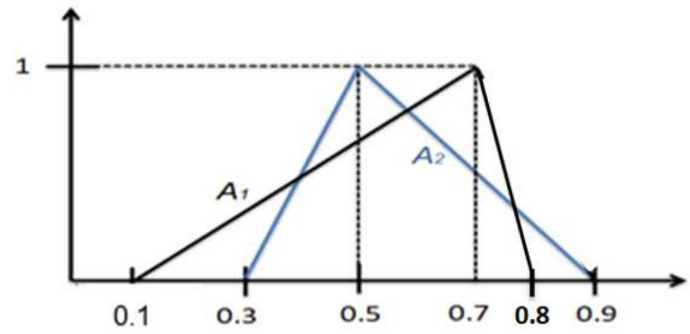

Figure 1. The triangular fuzzy number for financial asset returns A_1 and A_2

4.2. The expected financial asset return (Markowitz`s Theory)

According to Modern Portfolio Theory, the expected financial return "can be viewed as the historic average of a stock's return over a given period of time" (Mangram, 2013). The expected return for asset *i*, using the non-fuzzy number, is calculated by arithmetic mean, $E(R_{ai}) = \frac{1}{n}\sum_{j=1}^{n} R_j$, where R_j is the return of the asset *i* in *j* -time, when $j = \overline{1,n}$.

Definition 5: The expected value of the return for asset A_i is the medium value of the triangular fuzzy return $R_{ai} = (r_{ai} \quad r_{bi} \quad r_{ci})$ (Luengo, 2010):

$$E_f(R_{ai}) = \frac{r_{ai} + 4r_{bi} + r_{ci}}{6} = \frac{1}{6}(r_{ai} + r_{ci}) + \frac{2}{3} r_{bi}$$

where: $E_f(R_{ai})$ - triangular fuzzy expected return for asset *i*;

$r_{ai} \quad r_{bi} \quad r_{ci}$ - left, peak, right elements of triangular fuzzy return.

The expected return for the portfolio with A_i assets, where $i = \overline{1,n}$, is:

$$E_f(R_a) == \frac{1}{6}(r_{ai} + 4r_{bi} + r_{ci})$$

Example 2 (Continuing Example 1):

According to the Markowitz Theory, the expected non-fuzzy returns (means of the returns in the year N) for assets A_1 and A_2 are: 0.56 for A_1 and 0.58 for A_2.

The expected fuzzy rate return for the two assets A_1 and A_2 with financial returns: R_{a1} (0.1 0.7 0.8) and R_{a2} (0.3 0.5 0.9), considering the above formulas, are:

$$E_f(R_{a1}) = \frac{1}{6}(r_{a1} + r_{c1}) + \frac{2}{3}r_{b1} = \frac{1}{6}(0.1 + 0.8) + \frac{2}{3}0.7 = (\frac{1}{6}0.9 + \frac{2}{3}0.7) = 0.62$$

$$E_f(R_{a2}) = \frac{1}{6}(r_{a2} + r_{c2}) + \frac{2}{3}r_{b2} = \frac{1}{6}(0.3 + 0.9) + \frac{2}{3}0.5 = (\frac{1}{6}0.12 + \frac{2}{3}0.5) = 0.53$$

From this example, it can be observed that the non-fuzzy financial return and the triangular fuzzy return for asset A_1 have different values; the same goes for A_2. This difference comes from the fact that the non-fuzzy return does not consider the frequency of the return values in year N, while the triangular return considers the frequency of the peak value in year N and offers a higher weight of the most frequent return value. This means that the fuzzy expected return is more feasible based on trades history.

5. Modelling the financial risk with fuzzy triangular numbers

Financial risk "refers to the chance that the actual outcome (return) from an investment will differ from an expected outcome" (Omisore, 2012). In Markowitz Theory, the formula for financial risk is standard deviation from expected return, the square root of variance:

$$\sigma_i{}^2 = \frac{1}{n-1}\sum_{i=1}^{j}\left(R_j - E(R_i)\right)^2$$

Where: $E(R_i)$ - is the arithmetic mean for each asset *i*.

Definition 6: The risk of asset A_i, with the return: $R_{ai} = (r_{ai} \quad r_{bi} \quad r_{ci})$, is as follows (Boloș et al, 2019):

$$\sigma_i{}^2 = \frac{1}{4}\left[(r_{bi} - r_{ai})^2 + (r_{ci} - r_{bi})^2\right] + \frac{2}{3}\left[r_{ai}(r_{bi} - r_{ai}) - r_{ci}(r_{ci} - r_{bi})\right] + \frac{1}{2}\left(r_{ai}{}^2 + r_{ci}{}^2\right) - \frac{1}{2}E_f{}^2(R_{ai})$$

Example 4: Let the same assets from Example 1, have the same financial returns: R_{a1} (0.1 0.7 0.8) and R_{a2} (0.3 0.5 0.9). The non-fuzzy variances of these returns are: 0.46 for A_1 and 0.31 for A_2.

The triangular fuzzy variances for A_1 and A_2 are:

$$\sigma_{a1}{}^2 = \frac{1}{4}\left[(0.7 - 0.1)^2 + (0.8 - 0.7)^2\right] + \frac{2}{3}\left[0.1(0.7 - 0.1) - 0.8(0.8 - 0.7)\right] + \frac{1}{2}\left(0.1^2 + 0.8^2\right) - \frac{1}{2}(0.62)^2 = 0.207$$

$$\sigma_{a2}{}^2 = \frac{1}{4}\left[(0.5-0.3)^2 + (0.9-0.5)^2\right] + \frac{2}{3}\left[0.3(0.5-0.3) - 0.9(0.9-0.5)\right]$$
$$+\frac{1}{2}\left(0.3^2 + 0.9^2\right) - \frac{1}{2}(0.53)^2 = 0.159$$

6. Modelling the covariance with triangular fuzzy numbers

Covariance measures the relationship between the returns for one asset and returns for another. If the covariance is positive, the returns of the assets are positively related. If the covariance is negative, the return of the assets is negatively related. “It is necessary to avoid investing in securities with high covariances within themselves” (Markowitz,1952:89).
In Modern Portfolio Theory, the covariance is used for portfolio diversification. In order to reduce portfolio risk, an investor must include assets with negative covariances. By adding financial assets that have negative covariances, the risk of the portfolio is minimized.
The non-fuzzy formula for covariance is:

$$Cov(R_{a1}, R_{a2}) = \frac{1}{n-1}\sum_{1=j}^{n}(R_{1j} - E(R_{a1}))(R_{2j} - E(R_{a2}))$$

Definition 7: Considering the level sets of the returns for assets A_1 and A_2 :

$R_{a11}(\alpha) = (r_{b11} - r_{a11})\alpha + r_{a11}$;
$R_{a12}(\alpha) = r_{c11} - (r_{c11} - r_{b11})\alpha$, for financial return of asset A_1;
$R_{a21}(\alpha) = (r_{b21} - r_{a21})\alpha + r_{a21}$;
$R_{a11}(\alpha) = (r_{b21} - r_{a21})\alpha + r_{a21}$, for financial return of asset A_2;

the covariance between financial returns of assets A_1 and A_2 is:

$$Cov(R_{a1}, R_{a2}) = \frac{1}{4}[(r_{b11} - r_{a11})(r_{b21} - r_{a21}) + (r_{c11} - r_{b11})(r_{c21} - r_{b21})]$$
$$+\frac{1}{3}\{[(r_{a21}(r_{b11} - r_{a11}) + r_{a11}(r_{b21} - r_{a21})]$$
$$-[r_{c11}(r_{c21} - r_{b21}) + r_{c21}(r_{c11} - r_{b11})]\}$$
$$+\frac{1}{2}(r_{a11}r_{a21} + r_{c11}r_{c21}) + \frac{1}{2}E_f(R_{a1})E_f(R_{a2})$$

Example 5: The covariance between the returns of the assets, from first example, A_1 and A_2 is: 0.0119. That means that these two assets are positively related; when the return of first asset increases, the return of the second asset increases as well. On the other hand, when the first asset has a decreasing return, the return of the second asset follows the same pattern. The fuzzy covariance is determined as follows:

$$Cov(R_{a1}, R_{a2}) = \frac{1}{4}[(0.7-0.1)(0.5-0.3)+(0.8-0.7)(0.9-0.5)]$$
$$+\frac{1}{3}\{[(0.3(0.7-0.1)+0.1(0.5-0.3)]-[0.8(0.9-0.5)+0.9(0.8-0.7)]\}$$
$$+\frac{1}{2}(0.1\cdot 0.3+0.8\cdot 0.9)+\frac{1}{2}0.62\cdot 0.53=0.51$$

The fuzzy covariance is positively expressed in non-fuzzy number, because the multiplying and dividing arithmetical operations have real results, not triangular fuzzy results. The positive covariance represents a positive relationship between the two returns of the assets A_1 and A_2; it is the same positive result as the result of traditional covariance. In order to reduce the overall risk, the investors shoul not include these assets in the same portfolio.

7. Conclusion

In conclusion, the advantages and disadvantages of using fuzzy logic in modelling the indicators of Modern Portfolio Theory are summarized in the following SWOT Analysis:

Table 1. SWOT Analysis of the Portfolio Selection with Triangular Fuzzy Number

Strengths	Weaknesses
• Using fuzzy numbers - that are expressed in sets of real numbers, instead of using standard real numbers - is more reliable when the investor wants to estimate the risk or the gain for his portfolio, or for his future transactions; • Also, investment decision involves uncertainty. The expected return and the risk that the investor may assume are more accurate if they are expressed in sets of values, or triangular fuzzy number; • The membership function of the financial asset return or the membership function of the financial risk indicates the most frequent value in a given period and the degree of attaining the	• The expected return and risk are analysed based on the history of the investor`s trades. Even though the decision process is more flexible with the triangular fuzzy number, the investor cannot estimate the possibility to obtain the returns that are expressed as intervals. The risk of not attaining the past values remains. • The volatility of the financial market has a systematic risk and an unsystematic risk. The triangular fuzzy number does not consider the systematic risk. The values of the expected return can exceed the limits of the triangular fuzzy set.

other values of return. By using this component of the fuzzy number, the investor will be informed about the realization degree of the expected return values, analysing the frequency of the returns in the previous period; • The covariance considers the membership function of the returns for the assets; thus, it evaluates the correlation between the returns, given the trend of the returns.	
Opportunities	**Threats**
• Providing the possibility for the investor to estimate the probability of attaining the past returns, in the next period, this can be a direction for developing the Modern Portfolio Theory with the fuzzy number. • Extending the analysis period in order to obtain relevant results and therefore to help investors to estimate exactly the return and the risk for the portfolio. The longer the alaysis period – the more accurate estimation.	• If there are more assets in the portfolio, the calculation of Markowitz`s indicators (standard deviation, covariance, and correlation coefficient) will be more complicated.

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