ON THE IMPORTANCE OF ADAPTING UTILITY FUNCTIONS TO INVESTOR'S ATTITUDE TOWARDS RISK

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Abstract: *Within the current economic reality, investors are confronted daily with the problem of choosing between more investment options, when acting inside a risky business environment. Thus, the problem of choosing between the existing investment options becomes a multi-criteria problem, within which the investor must take into account not only to maximize the profit, but also to minimize the associated risk. Eventually, the primary aim of any investor is to maximize the utility felt by choosing a certain investment option. We leave from the assumption that an investment option considered to be best in terms of utility by a risk loving investor, will most certainly fail to satisfy a risk averse investor, as the two investors seek different opportunities. In this respect, we come to conclude that a decision making criterion that considers a certain utility function, should adapt the mathematical form of that utility function to fit the user's attitude towards risk. We consider a decision making criterion to be reliable only if able to maximize the investor's utility function by taking into account his attitude towards risk. Therefore, we will analyze the decision-making criterion under risky conditions proposed by Daniel Bernoulli by adapting its utility function to the investor's attitude towards risk. In this respect, we will firstly analyze the characteristics that a utility function should have to be fit for a certain investment profile, in mathematical terms. Following this analysis, we find that different profiles determine different characteristics and, therefore, different utility graphic shapes. Thus, we propose different functions, each of them proper for the investor's attitude towards risk, in order to expose and use Bernoulli's criterion. After analyzing each investment profile through its proper utility function, we come to the conclusion that our assumption regarding the difference in utility felt by a risk averse investor and a risk loving investor, was correct, as the best investment option is different for each investment profile.*

Key words: utility function, attitude towards risk, expected utility;

JEL Classification: D81

1. Introduction

The multitude of investment options available nowadays within the financial markets, the economic conditions and the fluctuations of every economic agent's activity leaves us the open issue of choosing between several investing options under risky or even uncertain conditions. This problem remained among the economists' research concerns, from early ages until the present time. Many decisional criteria have been created during time by several renowned scientists, each criterion being meant to bring something new and to improve the economic knowledge. Daniel Bernoulli has the great merit of pointing out the fact that every individual which acts as an investor, judges his actions not taking into consideration the price of the investment option available to him, but the utility felt by him after making his choice. The Expected Utility Theory created by Daniel Bernoulli changed the economic way of thinking forever, also succeeding to create the premises needed by later researcher to develop the theory of investors' attitude towards risk. Within the context of choice between several possible options, under risky conditions, in 1738, Daniel Bernoulli's researches led to the concepts of utility and marginal utility. Later, the concept

of marginal utility became the central point from which the Neoclassical Economic Theory emerged. The work of Daniel Bernoulli was based on the existing theories of that period regarding risk measurement, theories that had been developed by mathematicians over time and based on the absolute value that can be gained or lost. Those researchers had reached the following agreement:
"Expected values are determined by multiplying each possible gain by the number of its occurrence possibilities, and then dividing the sum of these products to the number of all possible cases, where, within this theory we insist over the cases that have the same probability of occurrence."
Bernoulli explains the meaning of this statement in the following way: if two individuals meet identical risks, they should expect equal value results. He challenges this idea, by giving an example to substantiate his opinion. His example emerges from a poor individual who obtains a lottery ticket. By keeping this lottery ticket, there were two possible outcomes: to win twenty thousand ducats or not to win anything. But the individual also had the possibility to sell the lottery ticket for the sum of nine thousand ducats. Bernoulli states that for this poor individual the rational solution would be to agree to sell the lottery ticket for nine thousand ducats, whereas from the point of view of a wealthy individual, the rational option would be to buy that ticket for the sum of nine thousand ducats. That's because winning a certain amount of money does not have the same importance for the two individuals. Through this example, Bernoulli proves wrong the mathematicians' conclusion on how to determine the expected value, suggesting instead, as a measure of value, the utility in exchange for the price. The price of a good is determined by the intrinsic qualities of the good in question, being equal for all individuals. However, the utility of that good is not the same for all individuals. The person who estimates the utility and the circumstances he find himself into influence the degree of utility felt. Thus, gaining an amount of one thousand ducats is more important for a poor individual than for a rich one, although both of them earn the same amount. Going through this reasoning, Bernoulli formulates the fundamental rule of measuring the risk to gain an amount of money through the utility associated to the probability of gaining:
"If the utility of each possible profit is multiplied by the number of its occurrence possibilities, and then the sum of these products is divided by the total number of possible cases, one will obtain an average utility and the profit corresponding to this utility will emphasize the value of the associated risk."
In 1948, Milton Friedman and Leonard J. Savage defined the connection between the investors' utility and their behavior, taking into consideration their attitude towards risk. Their researches pointed out the connection between the investors' attitude towards risk and the sign of the utility function. The utility function, in their opinion, doesn't always have a positive second derivative, the second derivative's sign depending on the investor's attitude towards risk.
Thus, for an individual who has risk aversion, the additional amounts of money that he can win have decreasing marginal utilities. The decreasing marginal utilities are explained by the fact that from the point of view of the individual with risk aversion, additional gains do not compensate for the undertaken risk. In other words, for such individuals, what may be lost as a result of the occurrence of the risk is much more valuable than what may be gained if the risk doesn't occur. So, for this kind of investors, the initial gains are important, the amounts of money initially won (even if they are small) bringing high marginal utility (and thereby determining an increasing trend of total utility function). As the individual has already accumulated some wealth, additional amounts won begin to lose importance and have negative marginal utility (leading to a downward trend of the total utility function). The switch from positive marginal utility to negative marginal utility is explained by the fact that for the individual, once he has reached a certain wealth, it becomes more important not to lose anything (to preserve the wealth gained) than undertaking any risk in the attempt to gain something.

In our opinion, the situation of the risk averse investor is identical to the situation of the poor individual obtaining a lottery ticket, presented in Daniel Bernoulli's theory. Just like the risk averse investor (for whom the initial amounts are more important, preferring to win small amounts than not to win anything until he accumulates a certain wealth), the poor individual poor in Bernoulli's example prefers achieving a certain wealth by selling the lottery ticket for nine thousand ducats, although its possession would maybe have brought twenty thousand ducats. He prefers therefore the sure but lower gain, over a big, but risky one.
In conclusion, the situation presented above describes an increasing utility function at first (due to positive initial marginal utility), which afterwards becomes decreasing (because the marginal utilities become negative). This is, in fact, the description of a concave function. A concave function is actually determined by a negative second order derivative. Therefore, it is concluded that if an investor is risk averse, the second derivative of his utility function is negative, the graph of the concave utility function being illustrated in figure 1.

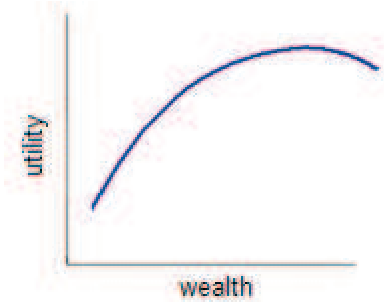

Figure 1: Utility function for risk averse investors
Source: simplified reproduction, made by the authors, after Friedman and Savage (1948)

On the other hand, if the investor has a preference for risk, the additional gains that he may achieve bring him increasing marginal utility. Increasing marginal utility is due to the fact that from the point of view of the risk preferring investor, additional gains which may increase his wealth compensate the undertaken risk. For him, initially gained small wealth is not as important as the large potential gains that he could still get if he keeps investing, even if these gains have a high associated risk. This kind of investor doesn't get any satisfaction from small gains or from a small wealth. In other words, for such an investor, what he may gain if the risk doesn't occur is much more valuable than what he may lose as a result of the occurrence of the risk. So, for the investor who prefers risk, initial gains are not important, the amounts of money gained initially bringing low marginal utility (thus leading to a decreasing trend of the total utility function). As the individual continuously increases its wealth, additional amounts won are becoming increasingly important, the marginal utility becoming positive (leading to an increasing trend of total utility function). This switch of the marginal utility from negative to positive is due to the fact that for the investor with preference for risk it's more important to keep trying to win and increase his wealth, than to remain with a low risk free wealth.
In our opinion, the situation of the risk preferring investor resembles the situation of the rich individual buying the lottery ticket, described in Daniel Bernoulli's theory. Just like the investor with preference for risk (for which the additional amounts gained are more important, preferring to risk losing a small fortune in hopes of obtaining a high gain), the rich individual in Bernoulli's example prefers to lose nine thousand ducats, (a small sum in his opinion), paying for the lottery ticket in hopes of winning a sum of twenty thousand ducats (and significantly increase his wealth), although this action is risky. He prefers therefore the higher gain, though risky, to the detriment of a small wealth, unsatisfactory from his point of view.

Thus, the situation described above is associated with a utility function that is decreasing in the beginning (because of the initial negative marginal utility), and afterwards becomes increasing (for the marginal utility becomes positive). All these facts describe, in fact, a convex function. A convex function is actually determined by a positive second order derivative. We, therefore, conclude that if an investor presents preference towards risk, the second derivative of the utility function is positive and the graph of the utility function is a convex one, as shown in figure 2.

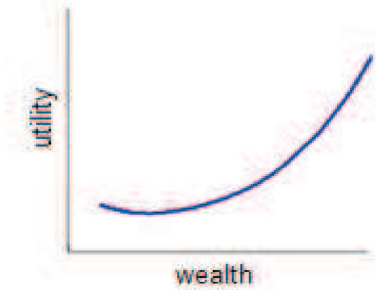

Figure 2: Utility function for risk preferring investors
Source: simplified reproduction, made by the authors, after Friedman and Savage (1948)

It is important to note that the point of minimum of the convex utility function marks the point where the earnings become interesting for the investor. Beyond this point, the risk preferring investor will feel motivated to invest in order to achieve large gains, interesting from his point of view.

2. Setting Assumptions

Taking into account all the facts described within the introduction part of this paper, we shall continue by establishing a set of assumptions regarding the utility functions proper for each investment profile.

We shall consider that the risk averse individual is, in fact, a poor individual concerned to achieve a certain wealth, even a low one. This individual does not want to lose anything that could affect his wealth and is not, therefore, willing to take any chances.

In order to illustrate his utility function, and to simplify the reasoning, we will choose a second degree utility function, depending on the wealth level, w:

$$\mathrm{U(w) = aw^2 + bw + c} \qquad (2.1)$$

Further, we will adapt this second degree function's parameters to the individual's investment profile.

The parameter c shows the level of utility felt by the individual when having no money. We will consider that nobody can fell any kind of utility if their wealth is null, so that for both cases described (risk aversion and risk preference), zero money will lead to zero utility.

$$\mathrm{f(0) = c = 0} \qquad (2.2)$$

Taking into consideration Friedman and Savage's description of the aversion towards risk, we know that the utility function in this case is a concave one. This means that the second derivative of the utility function must be negative.

$$\mathrm{U''(w) = 2a} \qquad (2.3)$$

Therefore, we come to conclude that $\mathrm{a} < 0$ (2.4).

The marginal utility is, in fact, mathematically, the first derivative of the utility function. In the case of the risk averse individual, the marginal utility is positive initially, and becomes negative after the targeted level of wealth has been reached. So, in mathematical terms, the first derivative of the utility function must be positive in the beginning and must become negative after the point of maximum (the targeted level of wealth being, in this case, the point of maximum). The first derivative of the utility function is presented below:

$$U'(w) = 2ax + b (2.5)$$

The wanted wealth and its corresponding utility level are, in fact, the Cartesian coordinates of the parabola's vertex (or turning point). So, the targeted wealth is $-\frac{b}{2a}, b \neq 0$ (2.6), b has to be different from 0, as the targeted wealth can't reasonably be 0. The maximum utility corresponding to it will be $-\frac{\Delta}{4a} = -\frac{b^2-4ac}{4a} = -\frac{b^2}{4a}$, as $c = 0$ (2.7).

The targeted wealth must be a positive value, so if $-\frac{b}{2a} > 0$ and $a < 0$, we conclude that $b > 0$ (2.8).

The maximum reached utility must, also, be a positive number, so $-\frac{b^2}{4a} > 0$ (2.9), which is correct as $b^2 > 0$ and $-4a > 0$ (2.10).

Therefore, we reach the following utility function for an individual that has aversion towards risk:

$$U(w) = aw^2 + bw \qquad (2.11),$$

whereas $a < 0$ and $b > 0$.

We will choose the same second degree utility function in order to illustrate the utility function of a risk preferring individual:

$$U(w) = aw^2 + bw$$

We have already agreed that both functions will be constant free.

Taking into account Friedman and Savage's description of the preference towards risk, we know that the utility function in this case is a convex one, meaning that the second derivative of the utility function must be positive.

We conclude, therefore, that $a > 0$ (2.12).

In the case of the risk preferring individual, the marginal utility is initially negative, becoming positive as the gains become bigger. So, in mathematical terms, the first derivative of the utility function must be negative at first, becoming positive after the point of minimum is reached (that point where the earnings become interesting).

The point where the gains become interesting for this individual is $-\frac{b}{2a}$, and the minimum utility corresponding to it will be $-\frac{\Delta}{4a} = -\frac{b^2-4ac}{4a} = -\frac{b^2}{4a}$, as $c = 0$.

The point where the gains become interesting must be a positive one, because the losses can't be interesting for anyone, so if $-\frac{b}{2a} > 0$ (2.13) and $a > 0$, we conclude that $b < 0$ (2.14).

Analyzing the vertex of this parabola, we see that, in the case of the preference towards risk, the minimum point in utility is a negative one. The rich individual is so disappointed by the small initial gains, that he sees them as damage. He doesn't need the small amounts of money wined; they don't fulfill his wishes, as he only wants to win big. As the winning becomes bigger and bigger, the rich, risk preferring individual becomes satisfied and feels their utility. The total utility function starts growing, eventually becoming positive. Solving the second degree equation $w^2 + bw = 0$, we find out the point where to total utility function becomes positive, that point being $-\frac{b}{a}$ (2.15).

Therefore, we reach the following utility function for an individual that has preference towards risk:

$$U(w) = aw^2 + bw \qquad (2.16),$$

whereas $a > 0$ and $b < 0$.

We conclude that in this case figure 2 does not capture the reality correctly. The case of the investor who prefers risk, is better illustrated in figure 3.

Figure 3: Utility function for risk preferring investors, as proposed by the authors
Source: graph made by the authors

3. Bernoulli's Decision Making Criterion, with the Utility Function Adapted to the Investor's Attitude Towards Risk

In order to exemplify Bernoulli's decision making criteria, taking into account the investor's attitude towards risk, we shall consider that the individual is confronted with three possible investing lotteries. The actions he may take are investing in either one of the three lotteries (L_1, L_2 or L_3). The possible outcomes of these actions could be not to win anything, win 150% of the invested amount or win 300% of the invested amount. The price of the lottery ticket will be taken into account. So, if the investor doesn't win anything, he will, in fact, lose the amount paid for the ticket. If he will win anything, his final gain will be calculated by subtracting the price of the ticket from the gain.

Price of L_1 is 5 monetary units (m.u.), price of L_2 is 25 m.u. and price of L_3 is 50 m.u. For example, if the investor chooses L_1 and doesn't win anything, he will, in fact lose 5 m.u., if he wins 150%, he will have 150% of 5 m.u. minus the price of the ticket (5 m.u.), so he will gain 2.5 m.u. If he wins 300%, he will have 300% of 5 m.u. minus the price of the ticket (5 m.u.), so he will have 10 m.u.

The informational matrix is presented below. In order to get results accurately describing the reality, we considered the possibilities of gain from actions taken by the individual (or from lotteries between which he can opt) to be equiponderated:

Table 1: The informational matrix

		Possible outcome		
		win 0%	**win 150%**	**win 300%**
probability		**0,(3)**	**0,(3)**	**0,(3)**
Lotteries	L_1	-5	2,5	10
	L_2	-25	12,5	50
	L_3	-50	25	100

Source: scenario proposed by the authors

We will consider that the profit or loss determined by this investment equals the available wealth of the individual.

Within this decision making criterion, every strategy is evaluated by its expected utility, as follows:

$$EU(L_i) = \sum_{j=1}^{n} p_j U(w_{ij}),\ i = \overline{1,m} \qquad (17)$$

where:

i from 1 to m is the number of actions that the individual can chose among;

j from 1 to n are possible outcomes which may occur;

L_i is the lottery number i that the investor can choose;

$EU(L_i)$ is the expected utility of action L_i for each possible outcome;

w_{ij} is the gain of generated by action i within the outcome j;
$U(w_{ij})$ is the utility associated to gain w_{ij};
p_j is the probability of occurrence of outcome j.

The actions available to the investor will be ordered ascending, according to their expected utility, in correspondence with the following equivalence relation:

$$a_p > a_q \Leftrightarrow EU(a_p) > EU(a_q) \quad (18)$$

The above relation signifies that a certain strategy a_p is preferred to strategy $a_q$only if the expected utility of profits generated by strategy a_p is superior to the expected utility of profits generated by strategy a_q.
The optimal solution of Bernoulli's criterion corresponds to that action for which the expected utility of obtainable profits is maximum.
The optimal solution = $\max\{EU(a_i)\}, i = \overline{1, m}$ (19)
We will next evaluate each of the three actions according to Bernoulli's criterion, through its expected utility. In this respect, we must choose a form of the utility function.
We will start with the case of the investor who has aversion towards risk. According to relations (2.1), (2.2), (2.4), (2.6), (2.8) and (2.10) and by attributing values to a and b accordingly, we can propose the following utility function for the risk averse individual:

$$U(w) = -w^2 + 20w \quad (20)$$

Under these circumstances, the expected utility (from the risk averse investor's point of view) for each possible action would be:

$$EU(L_1) = 0.(3) * \left(-(-5)^2 + 20 * (-5)\right) + 0.(3) * (-2.5^2 + 20 * 2.5) + 0.(3) * (-10^2 + 20 * 10) = 6.25$$

$$EU(L_2) = 0.(3) * \left(-(-25)^2 + 20 * (-25)\right) + 0.(3) * (-12.5^2 + 20 * 12.5) + 0.(3) * (-50^2 + 20 * 50) = -843.75$$

$$EU(L_3) = 0.(3) * \left(-(-50)^2 + 20 * (-50)\right) + 0.(3) * (-25^2 + 20 * 25) + 0.(3) * (-100^2 + 20 * 100) = -3875$$

By ordering the expected utilities increasingly, we will obtain the preference order according to Bernoulli's criterion:
L_1 is preferred to L_2 (as $EU(L_1) > EU(L_2)$).
L_2 is preferred to L_3(as $EU(L_2) > EU(L_3)$).
The optimal solution for an investor with risk aversion is, therefore, L_1.
It also can be easily notice that the optimal solution is the only action bringing a positive utility function to the risk averse investor. The other two possible actions are on the other hand too expensive for his taste and on the other hand to risky to be chosen by him. So, the theory presented above is proven by this example. The investor who has aversion towards risk prefers less risky actions and the cheapest investment options, being aware of the fact that these kinds of actions will only bring small profits.

We will continue by analyzing the case of the investor who presents preference towards risk. According to relations (2.1), (2.2), (2.10), (2.11), (2.12) and (2.13) and by attributing values to a and b accordingly, we can propose the following utility function for the risk preferring individual:

$$U(w) = w^2 - 20w \quad (21)$$

Under these circumstances, the expected utility (from the risk preferring investor's point of view) for each possible action would be:

$$EU(L_1) = 0.(3) * \left((-5)^2 - 20 * (-5)\right) + 0.(3) * (2.5^2 - 20 * 2.5) + 0.(3) * (10^2 - 20 * 10) = -6.25$$

$$EU(L_2) = 0.(3) * \left((-25)^2 - 20 * (-25)\right) + 0.(3) * (12.5^2 - 20 * 12.5) + 0.(3) * (50^2 - 20 * 50) = 843.75$$

$$\mathrm{EU}(\mathrm{L}_3) = 0.(3) * \left((-50)^2 - 20 * (-50)\right) + 0.(3) * (25^2 - 20 * 25) + 0.(3) * (100^2 - 20 * 100) = 3875$$

By ordering the expected utilities increasingly, we will obtain the preference order according to Bernoulli's criterion:
L_3 is preferred to L_2 (as $\mathrm{EU}(\mathrm{L}_3) > \mathrm{EU}(\mathrm{L}_2)$).
L_2 is preferred to L_1(as $\mathrm{EU}(\mathrm{L}_2) > \mathrm{EU}(\mathrm{L}_1)$).
The optimal solution for an investor with risk aversion is, therefore, L_3.
We notice that, in this case, the first possible action is the most unattractive for the risk loving investor. The possible outcomes are so small, that by choosing this action, he would feel negative utility. He feels that by choosing this action, he would lose his time, even if he wins. The investor presenting preference towards risk prefers risky actions and expensive investment options, expecting high profits and being ready to assume any loss.

4. Conclusions

Following this analysis, the great importance of adapting the utility function to the investor's investment profile emerges. If the investors' expected utility should have been analyzed without taking into account the form of the utility function, the results would not reflect the reality, misleading the user. This fact is primarily proved by the fact that there is no common optimal solution to apply to both investment profiles: risk preferring and risk averse investors.
Investors act differently within the market. Their decisions are driven by their own knowledge, emotions, wealth, etc. Every investment option is evaluated from the investor's point of view, taking primarily into consideration the utility that the investor hopes to get by making a choice. But the individuals live and take actions within different environments. This means that there aren't universally valid decision making criteria, and every existing criterion should specify what type of investor it addresses to or should be adapted to every attitude towards risk.
Usually the wealth level determines the attitude towards risk.
Some can't afford to take chances, because of their wealth level. They prefer to pay small amounts of money in order to invest, as these small amounts are affordable for them and if they lose, they want to be sure not to lose important amounts of money. These are the investors who present aversion towards risk. Their goal is to gain a certain wealth. They wish to win money without risking money, even if this means that the maximum winnable amount will also be small. For them, the loss of a big amount of money would be more important that the gain of a big amount of money. So they prefer to invest small hoping to win small.
Opposite to these individuals, there are the ones getting satisfaction from winning big amounts of money. For these investors, who dispose of a big wealth, the goal is not to reach a certain wealth anymore, they already have it. They are only satisfied by big amounts of money, if they win small they consider it a waste of time, fact that diminishes their utility. They wish to win big money, being disposed to risk money. But by risking money, they also expect big gains.
So, an investment option with a big expected utility from a risk averse investor's point of view, will not have the same expected utility when analyzed by a risk preferring investor. Their criteria are different, so it is logical that the results should be different.
Taking into account all these psychological aspects, it is clear that every type of investor should use proper decision making criteria, depending on their attitude towards risk. From our analyses displayed above, it is proven that the same investment options, with the same probability of occurrence will not be seen the same by two investors having different attitudes towards risk.

Therefore, every investment should be evaluated according to the investor's attitude towards risk. Only then the chosen option will be proper for the investor and the expected return will be realistic.

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