STABILITY, BIFURCATIONS AND CHAOS IN UNEMPLOYMENT NON-LINEAR DYNAMICS

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Abstract: *The traditional analysis of unemployment in relation to real output dynamics is based on some empirical evidences deducted from Okun's studies. In particular the so called Okun's Law is expressed in a linear mathematical formulation, which cannot explain the fluctuation of the variables involved. This paper deals with an introductive study about the role of non-linearity in the investigation of unemployment dynamics. The main idea is the existence of a non-linear relation between the unemployment rate and the gap of GDP growth rate from its trend. The macroeconomic motivation of this idea moves from the consideration of two concatenate effects caused by a variation of the unemployment rate on the real output growth rate. These two effects are concatenate because there is a first effect that generates a secondary one on the same variable. When the unemployment rate changes, the first effect is the variation in the level of production in consequence of the variation in the level of such an important factor as labour force; the secondary effect is a consecutive variation in the level of production caused by the variation in the aggregate demand in consequence of the change of the individual disposal income originated by the previous variation of production itself. In this paper the analysis of unemployment dynamics is carried out by the use of the logistic map and the conditions for the existence of bifurcations (cycles) are determined. The study also allows to find the range of variability of some characteristic parameters that might be avoided for not having an absolute unpredictability of unemployment dynamics (deterministic chaos).*

Keywords: *Unemployment; Economic growth; Logistic map.*

JEL classification: *E24;C02.*

1. Introduction

The phase of the economic cycle has strong implications on the employment levels of a Country through different channels. Business decisions on investment and employment depend on several socio-economic variables. The aggregate demand influences the level of utilization of the productive factors, and this in the long term also means investment decisions and capital accumulation. The capital incorporates technology, affects the productivity of workers and contributes to the formation of the human capital. Furthermore the transfer of labour force from a declining sector to another may be a thrust which affects its size and its allocation.
Structural changes in the economic environment can cause profound changes in employment that may prevent from returning to levels recorded before a downturn.

In this case the potential public support, in addition to its being a guide for the general upturn, can move towards the retraining of workers or the facilitation of their transition from a declining industry to a rising one.
Among the instruments to analyze these complex relationships, there is the quantitative relation between the unemployment rate and the *GDP* gap studied by the economist Arthur Melvin Okun (1962) for the U.S. economy in the 60s.
The so-called *Okun's Law* originated from the observation of some regularity in the dynamics of specific economic variables. It was conceived by the economist Okun in 1962, and represents an instrument to compare the current output and the potential *GDP*, starting from the difference between the gap of current unemployment rate and its natural level.
The output gap, that is the difference between the current *GDP* and the potential one, may be positive or negative. In the first case the aggregate demand is greater than the aggregate supply, and this situation tends to increase inflation. In the negative case the economic system operates at a lower level than the optimal one. Okun observed an empirical evidence of the U.S. data, considering how the higher level of current unemployment respect to the natural one, could be considered a cause of an increasing gap between the current level of *GDP* and the potential one (trend). The *Okun coefficient* has been estimated several times in economic studies adopting the different reformulations of the original relationship (see, among many contributions, Smith, 1975; Gordon, 1984; Knoester, 1986; Kaufman, 1988; Prachowny, 1993 and Weber, 1995). Initially it was verified that an increasing of unemployment rate by 1%, originates a drop of 2-3% of the real *GDP* from its potential value. The economic explanation for this empirical regularity is due to the fact that, during periods in which there is expansion of output, more workers are employed or work hours are extended.
The *Okun's Law* can therefore be seen as an instrument of economic policy to control the unemployment level and the trend of *GDP*. Of course, the employment level depends on characteristics of the national system and the local labour market and then the value for other Countries may be different from the final effect in the United States. The aim of bringing unemployment close to the natural level could be reached by a support to the aggregate demand, also by public aids that could sustain the output level. With the increasing of output level, it becomes necessary to increase the number of the labour force in the system. Obviously the number of newly employed workers must be greater than the natural variation of population, because over time, thanks to knowledge and new technologies, the productivity of labour increases, implying the decrease of the number of workers necessary for an unchanged level of output.
Economic studies (see Silvapulle et al., 2004) have highlighted an asymmetry in the relation of Okun. The positive turns in the economic cycle are less influential on unemployment respect the negative ones. This condition should be valid in all the Countries like the U.S. that have a labour market with few restrictions that implies the possibility of a rapid decrease of the labour force.
Most recent determinations of the coefficient of Okun have confirmed the trend found in the 60s from empirical regularities, at least for U.S. and Canada, while the relationship is weaker in Europe and Japan (see, for an analysis on *G7* Countries, Moosa, 1997). In many European Countries labour regulation and the role of trade unions prevent firms from adapting the number of workers promptly to the needs dictated by the economic cycle.

The following chart shows the relationship between one-period (annual) changes in *GDP* and in the unemployment rate in the *Euro Area* (17).

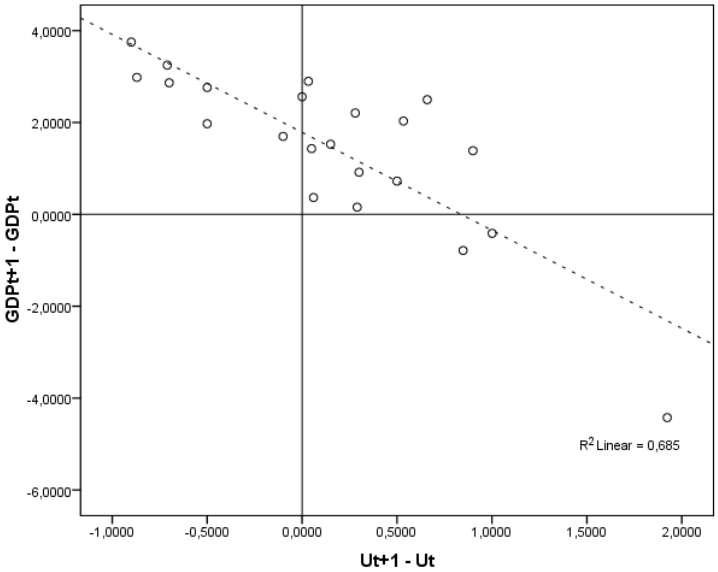

Figure 1: Relationship between one-period (annual) changes in *GDP* rate and in the unemployment rate for the *Euro Area* (17), 1992-2013* (*estimates)
Source: Our elaborations on *IMF* data

Even by the comparison between the annual variations of the two variables is clear the inverse relationship. This evidence is confirmed over time, though with obvious distinctions, and it is more evident in the largest economic contexts such as the *U.S.*, the *E.U.*, the *Euro Area* and the *G7* group, nonetheless remaining valid even in the single national contexts (see, among others, Mattoscio and Odoardi, 2012).

2. Unemployment non-linear dynamics and real output growth

The dynamic model that funds the analysis contained in this paper moves from a generalized version of *Okun's Law* that links the behaviour of real output growth and changes in unemployment over time. In this generalized form *Okun's Law* states that the unemployment rate *u* declines by γ percentage point for every 1 percentage point of annual real output growth *g* above trend. But the same law establishes that, if output growth were 1 percent below trend, unemployment would rise γ percentage point.
A mathematical form of this version of *Okun's Law* can be expressed in the following way (Dornbush and Fisher, 1985, p. 483):

$$u_{t+1} = u_t - \gamma\,(g_{t+1} - g_{t+1}{}^{e}) \qquad (1)$$

where u_t is the unemployment rate at time *t*; g_{t+1} and $g_{t+1}{}^{e}$ respectively are the growth rate of real output at time *t+1* and the expected one (trend); γ is a real positive

number (parameter).
The *Okun's Law* is an empirical relation between growth and unemployment, but its generalized form can be integrated by a theoretic hypothesis about the non-linearity of the relation itself.
In fact it has much significance to hypothesize the dependence of real output growth gap ($g_{t+1} - g_{t+1}^e$) on unemployment of the previous period as a quadratic function, in consequence of the presence of two interconnected effects on real output growth dynamics coming from a change in unemployment rate. We think to the double effect (direct and indirect) of a variation in unemployment rate on the real output growth: a direct effect, linked to technological aspects, is the variation in the level of production in consequence of the variation in the level of the labour factor; an indirect effect is the change in the level of production in consequence of the variation in the aggregate demand due to the different individual disposal income caused by the previous variation of production itself.
The analytical form of this type of quadratic relation can be written in the following way, according to the double effect described above and considering the algebraic signs of the coefficients α and β not predetermined:

$$g_{t+1} - g_{t+1}{}^{e} = -\alpha\, u_t - \beta\, u_t^2 \qquad (2)$$

(the negative signs in the second member of this equation are opportune for the following calculations).
By the simultaneous consideration of equalities (1) and (2) it is possible to obtain the following dynamic equation of the time path of the unemployment rate:

$$u_{t+1} = (1 + \alpha\, \gamma)\, u_t + \beta\, \gamma\, u_t^2 \qquad (3)$$

If we use an opportune linear transformation of variable u_t in (3), we can write an equation equivalent to (3) itself, that acquires the analytical form of the logistic map (May, 1976).
The transformation is:

$$u_t = -[(1 + \alpha\, \gamma) / (\beta\, \gamma)]\, U_t \qquad (4)$$

The logistic map so obtained is:

$$U_{t+1} = (1 + \alpha\, \gamma)\, U_t\, (1 - U_t); \qquad (5)$$

$(1 + \alpha\, \gamma)$ is the characteristic parameter of the map itself and equation (5) is the time path equation of the supporting variable U_t.

3. Steady states, stability analysis and conditions to avoid chaos

The discussion of the possible time paths of the variable U_t and consequently of the variable u_t that expresses the unemployment rate, with respect to the variation of the characteristic parameter $(1 + \alpha\, \gamma)$ leads to the conditions of the existence of bifurcations and periodic cycles and therefore to those related to the onset of deterministic chaos.
These latter in particular are important because they allow to know how to "avoid

the chaos" that is inherent in the dynamics of the system. This result is consistent with a very true statement: chaos must be "known" in order to "be avoided", since it is a phenomenon more close to everybody than one might think.
The first important considerations are to be referred to the conditions of existence of a stable steady state: the convergence of the unemployment rate to a stable steady state is conditioned by the variability of the parameter $(1 + \alpha\ \gamma)$ in the open set of real numbers (1,3), according to the stability analysis that is usually made in relation to the logistic map. On the other hand, for the existence of a stable steady state, the same condition, interpreted as the double inequality $1 < (1 + \alpha\ \gamma) < 3$, leads to impose that the parameter γ should vary in the open set of real number $(0, 2/\alpha)$, that is: $0 < \gamma < (2/\alpha)$. For the coherence of the algebraic signs in this inequalities, α would be positive, since γ is a positive parameter. The sign of β would be either positive or negative.
Analyzing the equation (3), that is the time path of the variable u_t, there are two values of the unemployment rate that could be interpreted as steady states: $u^*_1 = 0$ and $u^*_2 = -\alpha/\beta$, both of them determined by imposing $u_{t+1} = u_t$ in (3) itself. The two potential steady states so determined coincide with the values in which the gap $(g_{t+1} - g_{t+1}{}^{e})$ would be equal to zero, in consequence of the relation expressed by equation (2).
But $u^*_1 = 0$ (corresponding to $U^*_1 = 0$) must not be considered, being the equation (3) an identity in this value. We observe the coherence of this mathematical result with the economic significance: the steady state corresponding to an unemployment rate equal to zero is a trivial result.
Moreover $u^*_2 = -\alpha/\beta$, corresponding to $U^*_2 = [(\alpha\ \gamma) / (1 + \alpha\ \gamma)]$, is the only steady state to be considered; it exists and is stable if $0 < \gamma < (2/\alpha)$, according to the general results of the stability analysis of the logistic map.
Our analysis becomes more useful in relation to the aim of avoiding chaos that is the aim of avoiding the absolute unpredictability of unemployment dynamics. We summarize the main results by referring to three different ranges of values of the characteristic parameter γ of the logistic map. Another useful result of our study is the possibility of evicting cycles in unemployment rate dynamics.
When $(2/\alpha) \leq \gamma < (2.4495\ldots/\alpha)$ the time path of the variable U_t, and consequently of the unemployment rate u_t, undergoes a first bifurcation (2-frequency cycle).
When $(2.4495\ldots/\alpha) \leq \gamma < (2.5699\ldots/\alpha)$ bifurcations of the time path of the variable U_t, and consequently of the variable expressing the unemployment rate u_t, successively double themselves (2^n-frequency cycles).
Finally, when $(2.5699\ldots/\alpha) \leq \gamma \leq (3/\alpha)$, the time path of the variable U_t, and consequently of the variable expressing the unemployment rate u_t, evolves with the presence of cycles characterized by "infinite frequency", that is with the presence of the absolute uniqueness of any value of the rate itself, whose evolution "appears" stochastic, but it is only absolutely disordered, in other words the deterministic chaos occurs. The last closed set of real numbers should be "avoided", and consequently it should be avoided that the parameter γ of the generalized form of Okun's Law (1) varies in the interval of real numbers: $[(2.5699\ldots/\alpha), (3/\alpha)]$.

4. Graphical representations

The discussion about the numerical consequences of the variability of the parameter γ can be visualized by a simple graphical representation.

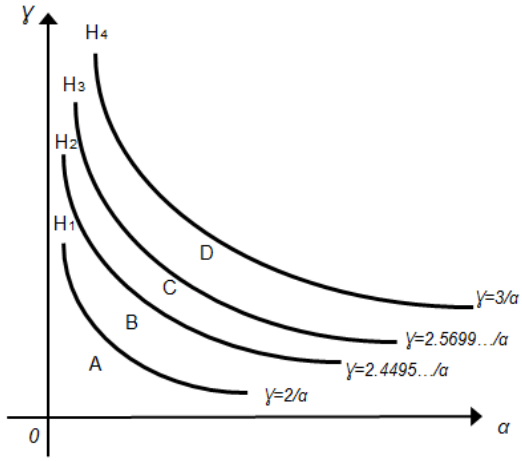

Figure 2: Sets of (α , γ) pairs.

The four hyperbola branches in the Figure 2 correspond to the four cases related to the following four equalities:

$$\gamma = 2/\alpha\ ,\ \ \gamma = 2.4495\ldots/\alpha\ ,\ \ \gamma = 2.5699\ldots/\alpha\ ,\ \ \gamma = 3/\alpha\ ;$$

these hyperbola branches form four regions in the first quarter, named *A*, *B*, *C*, *D*.
If (α , γ) belongs to the open region *A*, there is a potential convergence to the stable steady state $u^*_2 = -\alpha/\beta$ that is equivalent to $U^*_2 = [(\alpha\ \gamma) / (1 + \alpha\ \gamma)]$.
If (α , γ) belongs to the open region *B*, there is the first bifurcation and a 2-cycle.
If (α , γ) belongs to the open region *C*, bifurcations double over time and there are cycles characterized by increasing periodicity.
If (α , γ) belongs to the open region *D*, there is the onset of deterministic chaos.
If (α , γ) belongs to the branch H_1, there is one bifurcation; if (α , γ) belongs to the branch H_2, there is a first doubling of bifurcations; if (α , γ) belongs to the branches H_3 and H_4, there is deterministic chaos.

The following four graphs represent different cases corresponding to four values of the parameter $(1 + \alpha\ \gamma)$. In these graphical representations the axis variable is time and the ordinate is the supporting variable U_t.

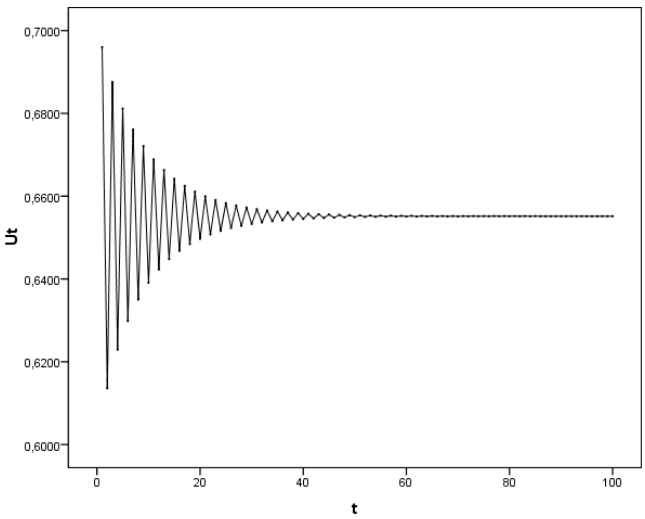

Figure 3: *$U_0 = 0.4$, $U_1 = 0.696$* and *$(1 + \alpha\,\gamma) = 2.9 \leftrightarrow \gamma = 1.9 / \alpha$*
(convergence to the only stable steady state $U^* = [(\alpha\,\gamma) / (1 + \alpha\,\gamma)] = 0.655$)

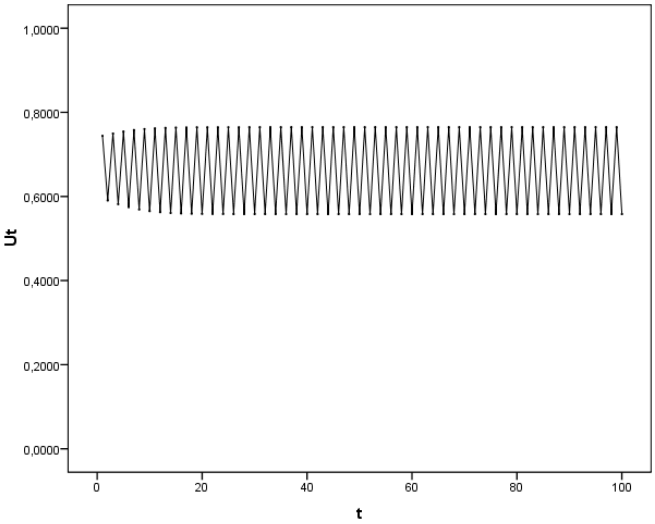

Figure 4: *$U_0 = 0.4$, $U_1 = 0.744$* and *$(1 + \alpha\,\gamma) = 3.1 \leftrightarrow \gamma = 2.1 / \alpha$*
(one bifurcation – a cycle of periodicity 2)

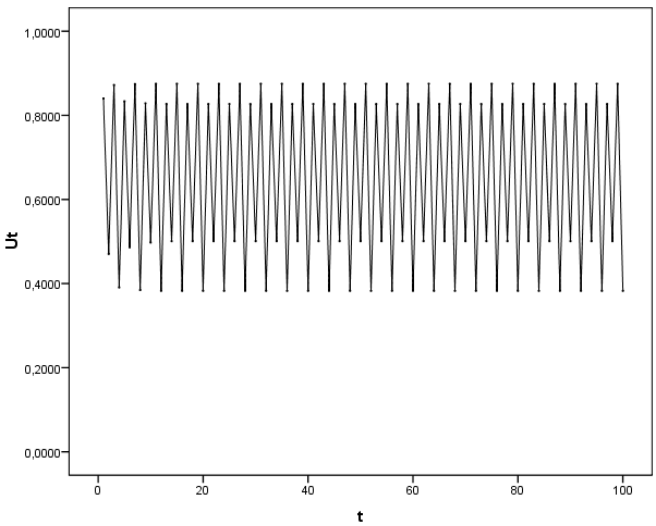

Figure 5: *$U_0 = 0.4$, $U_1 = 0.84$* and *$(1 + \alpha\ \gamma) = 3.5 \leftrightarrow \gamma = 2.5 / \alpha$*
(a first doubling of bifurcations)

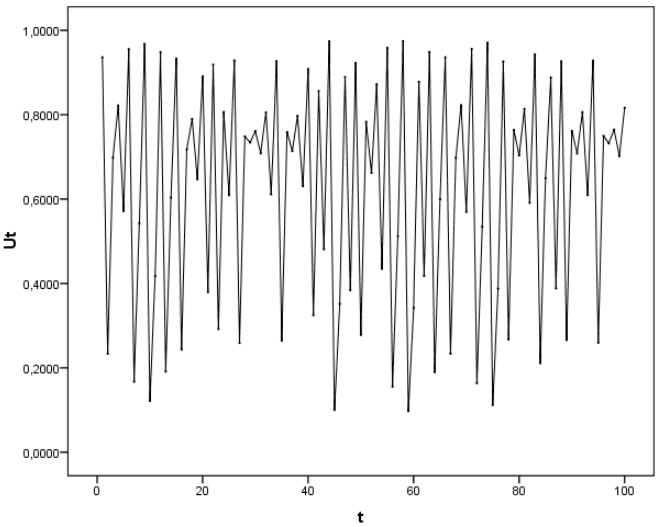

Figure 6: *$U_0 = 0.4$, $U_1 = 0.936$* and *$(1 + \alpha\ \gamma) = 3.9 \leftrightarrow \gamma = 2.9 / \alpha$*
(deterministic chaos)

The main contribution contained in this paper is the proposal of a method for reviewing *Okun's Law*, moving from the observation of real data that often show a non-linear relation between the GDP growth rate gap and the unemployment rate of the previous period, according to the hypothesis expressed by equation (2). The following graph represents this type of relation for European aggregate data.

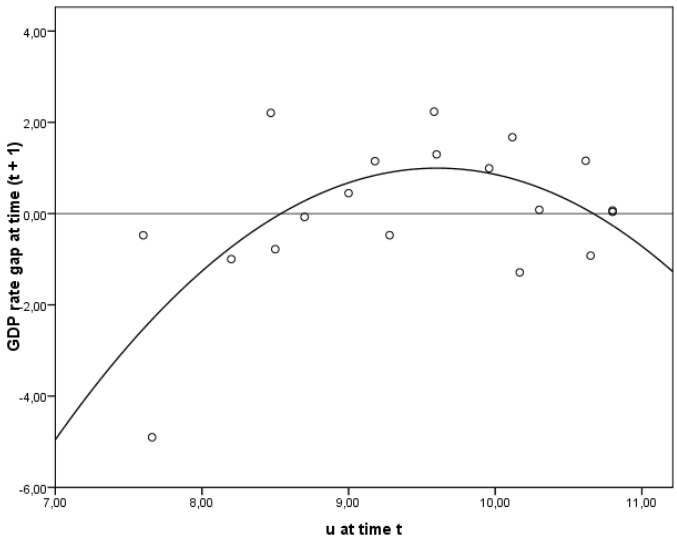

Figure 7: Relationship between *GDP* rate gap and the unemployment rate of the previous period for the *Euro Area* (17), 1994-2012
Source: Our elaborations on *IMF* data

5. Conclusions

The study contained in this paper is the theoretic step of the analysis aimed to formalize the relation between unemployment rate and real output growth.
The main contribution of this work is the method used to examine this link.
The analytical approach moves from the idea of the existence of an economically significant non-linear relation between the gap $(g_{t+1} - g_{t+1}{}^{e})$ and the unemployment rate.
Another important result of this paper is the individuation of conditions to avoid the onset of the unpredictability of the unemployment rate. The relevance of these conditions is due to the following equivalence: unpredictability is equivalent to uncontrollability because of the total absence of information about the future value of the variable to be controlled.
The study itself has to be completed by the econometric determination of the coefficient α and β of the quadratic relation (2) which expresses the dependence of the GDP growth rate gap on the unemployment rate of the previous period.

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