Partial differential equation, parabolic Black-Scholes type:

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0
\]

is used in evaluating equity options, that paying constant and continue dividends or in evaluate options in which interest rate, volatility and dividend are dependent on time.

**Keywords:** stocks, asset-support, options (call and put), contracts (futures and forward), portfolio, model

**JEL Classification:** G12, G13

### 1.1 Black-Scholes equation

Further we intend to Black-Scholes equations, capitalizing above the strategy used by Paul Wilmott:

Be \( \pi = V(S;t) - \Delta S \)

(1)

a portfolio consists of option, with value \( V = V(S;t) \)

and quantity: \( \Delta S \) (ie: „short” position, for asset-support: \( S \)).

Also, consider the value of „\( \Delta \)“:

\[ \Delta = \frac{\partial V}{\partial S} \quad (\text{random component: } \Delta(„\text{delta}“)) \]

(2)

Replaced, value of „\( \Delta \)“:

\[ \Delta = \frac{\partial V}{\partial S} \iff \Delta - \frac{\partial V}{\partial S} = 0, \]

in his expression: \( d\pi \), we obtain:

\[ d\pi = \left( \frac{\partial V}{\partial t} dt + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt + \left( \frac{\partial V}{\partial S} - \Delta \right) dS \right) \]

(3)

or equivalent:
\[ d\pi = \left( \frac{\partial V}{\partial t} dt + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt \]  
\text{(4)}

(ie: „increase” of \( \pi \) portofolio, free of risk)

But the variation of \( \pi \), ie: \( d\pi \) can write (using the nonarbitration principle) in the form:

\[ d\pi = \pi dt \]  
\text{(5)}

Conclusion:

Replaced the relations: (1), (2) and (4) \( \rightarrow (5) \), we obtain:

\[ \left( \frac{\partial V}{\partial t} dt + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt = r \left( V - S \frac{\partial V}{\partial S} \right) dt : dt \neq 0 \]  
\text{(6)}

\[ \leftrightarrow \quad \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = r \left( V - S \frac{\partial V}{\partial S} \right) \]  
\text{(7)}

Ordering the terms in relation (29°), we obtain finally equation seeking, respectively:

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} - r V = 0 \text{ (Black-Scholes equation)} \]  
\text{(8)}

1.2 Interpretation of the notions:

\( O_1 \) ) Black-Scholes equation (deduced above), was written for the first time in this form, in 1969, but published 4 years later, in 1973, when Fischer Black and Myron Scholes proved the correctness of this equation.

\( O_2 \) ) Black-Scholes equation models a very important financial and economic phenomenon: the assessment of stock options.

\( O_3 \) ) The Black-Scholes equation, there are very important variables and parameters, respectively:

- \( S = S(\mu; \sigma) \rightarrow \text{stochastic variable (is the value of the asset-holder)} \)

Where:

\[ \mu = \text{drift (or unexpected return)} \]

\[ \sigma = \text{asset volatility} \]

\[ t = \text{time (deterministic variable)} \]

\[ r = \text{interest rate (without risk)} \]

\[ V = V(S; t) \rightarrow \text{call option value (option to purchase)} \]
Terms: $\frac{\partial V}{\partial t}; \frac{\partial V}{\partial S}$ and $\frac{\partial^2 V}{\partial S^2}$, which appeared in Black-Scholes equation, is: „indicators of sensitivity” to change the option value „V”, respectively:

$$\Delta = \frac{\partial V}{\partial S} \rightarrow \text{pointer „delta”, stands for:}$$

„percentage of modified (percentage of modified (\(\partial V\)) option value \(V\), depending on the value of asset-support (\(\partial S\))”

or even: \(\Delta = \frac{\partial V}{\partial S} \rightarrow „hedge \ delta” \) useful (theoretical and practical) to eliminate the risk of a portfolio;

\(\theta = \frac{\partial V}{\partial t} \rightarrow \text{pointer „theta”, is:}$$

„modified of option price (\(\partial V\)), depending on the variation of time (\(\partial t\))”

\(\Gamma = \frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial S} \left( \frac{\partial V}{\partial S} \right) \rightarrow \text{pointer „gama”, is}$$

„sensitivity (excessive sensitivity) of \(\Delta = \frac{\partial V}{\partial S}\), to price variations of asset-support \(S\)”

Finally, in the Black-Scholes equation, deduced previously, appear two terms: \(rS \frac{\partial V}{\partial S}\) and \(rV\), where:

- term: \(rS \frac{\partial V}{\partial S}\), is the convection term of equation;
- term: \(rV\), is the reaction term (which balances the first term of the equation \(\frac{\partial V}{\partial t}\)).

In conclusion, bringing together the 4 (four) terms:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$

terms characterized briefly, above, we obtain Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$

equivalent to a special equations of fluid mechanics, type: „reaction-convection-diffusion”.

1.3 Black-Scholes equation solving

General overview:

Method for solving Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \quad (9)$$

aimed at obtaining the solution (exact) for the partial differential equation, parabolic type.
In literature, there are two broad categories of methods of solving this equation (ie: this type of partial differential equations).

- analitic methods
- numeric methods.

The analytical methods we can mention:

Method I: Transforming equation (8), in an equation with constant diffusion coefficient, using the change of variable:

\[ V(S; t) = e^{ax + \beta t} \cdot U(x; \tau) \]

In finally, „function”/solution \( U(x; \tau) \) satisfy the basic equation of diffusion:

\[ \frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2} \tag{10} \]

(equation easier to solve than Black-Scholes equation (9)).

Method II, using Green function:

\[ V(S; t) = \frac{e^{-r(T-t)}}{\sigma S \sqrt{2\pi(T-t)}} e^{\frac{\left[ \log\left(\frac{S}{S^*}\right) + (r - \frac{1}{2}\sigma^2)(T-t)^2 \right]}{2\sigma^2(T-t)}} \tag{11} \]

Observations:

1) In the limit (when \( t \rightarrow T \), and \( T-t \rightarrow 0 \) and \( e^{-r(T-t)} \rightarrow e^s = 1 \) function (derivative I of \( V \), given by (11) becomes Dirac delta function, ie:

„A function that is one (zero) everywhere except one point where it is infinite, so that”:

\[ \int V(S; t) dt = 1 \tag{12} \]

2) Function given by (11) is called Green function, which by integration, becomes:

\[ V(S; t) = \frac{e^{-r(T-t)}}{\sigma \sqrt{2\pi(T-t)}} \int_0^\infty e^{\frac{\left[ \log\left(\frac{S}{S^*}\right) + (r - \frac{1}{2}\sigma^2)(T-t)^2 \right]}{2\sigma^2(T-t)}} \cdot Payoff(S) \frac{dS'}{S} \tag{13} \]

where:

\( Payoff(S) = V(S; T) \rightarrow \text{ie: “profit/or loss for } S \text{ is equal to the option } V, \text{ at time } t=T” \tag{14} \)

There are other analytical methods to assess Black-Scholes equation, for example:

Fourier and Laplace transformations.

The methods numeric at solving partial differential equations, parabolic type (that is, and the equation: Black-Scholes), we can mention:

Finite difference method with the following:

- explicit finite difference method and
- implicit finite difference method

Very popular in the literature (to category: „numerical methods”) is the **Crank-Nicolson method**, which performs: „mean between the explicit and implicit method”.

**Explicit method**, applied Black-Scholes equation, to implement Black-Scholes model is applied to options: call and put European-style (or American type).

**Default method**, Crank-Nicolson equation applied to options: call and put European type, is more difficult to program, but has greater precision.

**Final conclusion:**

Black-Scholes Equation \( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \) represents a partial differential equation, parabolic type which corresponds to the Black-Scholes Model published by the American economists Fischer Black and Myron Scholes in 1973. This model may be used with success in evaluating equity options.

**References:**