„BLACK-SCHOLES MODEL USED TO EVALUATE STOCKS OPTIONS”

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Partial differential equation, parabolic Black-Scholes type:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0$$

is used in evaluating equity options, that paying constant and continue dividends or in evaluate options in which interest rate, volatility and dividend are dependent on time.

Keywords : stocks, asset-support,options (call and put), contracts (futures and forward), portofolio, model

JEL Classification: G12, G13

1.1 Black-Scholes equation

Further we intend to Black-Scholes equations, capitalizing above the strategy used by Paul Wilmott:

Be $\pi=V(S;t)-\Delta S$ (1)

a portfolio consists of option, with value V=V(S;t)
and quantity: ΔS (ie: „short” position, for asset-support: S).

Also, consider the value of „Δ”:

$$\Delta=\frac{\partial V}{\partial S} \quad \text{(random component: Δ(„delta”))} \qquad (2)$$

Replaced, value of „Δ”: $\Delta=\frac{\partial V}{\partial S}\Leftrightarrow \Delta-\frac{\partial V}{\partial S}=0,$

in his expression: dπ, we obtain:

$$d\pi=\left(\frac{\partial V}{\partial t}dt+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt+\left(\underbrace{\frac{\partial V}{\partial S}-\Delta}_{0}\right)dS \qquad (3)$$

or equivalent:

$$d\pi = \left(\frac{\partial V}{\partial t} dt + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt \qquad (4)$$

(ie: „increase" of π portofolio, free of risk)

But the variation of π, ie: dπ can write (using the nonarbitration principle) in the form:

$$d\pi = r\pi dt \qquad (5)$$

Conclusion:

Replaced the relations: (1), (2) and (4) → (5), we obtain:

$$\left(\frac{\partial V}{\partial t} dt + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt = r \left(V - S \frac{\partial V}{\partial S} \right) dt \Big| : dt \neq 0 \qquad (6)$$

$$\leftrightarrow \quad \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = r \left(V - S \frac{\partial V}{\partial S} \right) \qquad (7)$$

Ordering the terms in relation (29`), we obtain finally equation seeking, respectively:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \quad \text{(Black-Scholes equation)} \qquad (8)$$

1.2 Interpretation of the notions:

O_1) Black-Scholes equation (deduced above), was written for the first time in this form, in 1969, but published 4 years later, in 1973, when Fischer Black and Myron Scholes proved the correctness of this equation.

O_2) Black-Scholes equation models a very important financial and economic phenomenon: the assessment of stock options.

O_3) The Black-Scholes equation, there are very important variables and parameters, respectively:

• S=S(μ;σ) → stochastic variable (is the value of the asset-holder)

Where:

μ= drift (or unexpected return)
σ= asset volatility } parameters associated asset price „S"

t = time (deterministic variable)
r = interest rate (without risk)
V = V(S;t) → call option value (option to purchase).

O_4) Terms: $\frac{\partial V}{\partial t}; \frac{\partial V}{\partial S}$ *and* $\frac{\partial^2 V}{\partial S^2}$, which appeared in Black-Scholes equation, is: *„indicators of sensitivity”* to change the option value „V”, respectively:

$\Delta = \frac{\partial V}{\partial S} \rightarrow$ pointer „delta”, stands for:

„percentage of modified (percentage of modified (∂V) option value V, depending on the value of asset-suport (∂S)”

or even: $\Delta = \frac{\partial V}{\partial S}$ → „hedge delta” useful (theoretical and practical) to eliminate the risk of a portfolio;

$\theta = \frac{\partial V}{\partial t}$ → pointer „theta”, is:
„modified of option price (∂V), depending on the variation of time (∂t)”

$\Gamma = \frac{\partial^2 V}{\partial S^2} \stackrel{\text{def}}{=} \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial S}\right)$ → pointer „gama”, is
„sensitivity (excessive sensitivity)of $\Delta = \frac{\partial V}{\partial S}$, to price variations of asset-support S”

O_5) Finally, in the Black-Scholes equation, deduced previously, appear two terms: $rS\frac{\partial V}{\partial S}$ and rV, where:

- term: $rS\frac{\partial V}{\partial S}$, is the *convection* term of equation;
- term: rV, is the *reaction* term (which balances the first term of the equation $\frac{\partial V}{\partial t}$).

O_6) In conclusion, bringing together the 4 (four) terms:

$$\frac{\partial V}{\partial t};\ \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2};\ rS\frac{\partial V}{\partial S}\ \text{şi -rV;}$$

terms characterized briefly, above, we obtain Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

equivalent to a special equations of fluid mechanics, type: „reaction-convection-diffusion”.

1.3 Black-Scholes equation solving

General overview:

Method for solving Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \qquad (9)$$

aimed at obtaining the solution (exact) for the partial differential equation, parabolic type.

In literature, there are two broad categories of methods of solving this equation (ie: this type of partial differential equations).

- analitic methods
- numeric methods.

The analytical methods we can mention:

Method I: Transforming equation (8), in an equation with constant diffusion coefficient, using the change of variable:

$$V(S;t) = e^{\alpha x+\beta\tau} \cdot U(x;\tau)$$

In finally, „finction"/solution U(x;τ) satisfy the basic equation of diffusion:

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2} \qquad (10)$$

(equation easier to solve than Black-Scholes equation (9)).
Method II, using Green function:

$$V^{'}(S;t) = \frac{e^{-r(T-t)}}{\sigma S^{'}\sqrt{2\pi(T-t)}} e^{\frac{\left[\log\left(\frac{S}{S^{'}}\right)+(r-\frac{1}{2}\sigma^2)(T-t)^2\right]}{2\sigma^2(T-t)}} \qquad (11)$$

Observations:

1) In the limit (when t→T, and T-t→0 and $e^{-r(T-t)}$→e°=1) function (derivative I of V, given by (11) becomes *Dirac delta function*, ie:

„A function that is one (zero) everywhere except one point where it is infinite, so that":

$$\int V^{'}(S;t)dt = 1 \qquad (12)$$

2) Function given by (11) is called *Green function*, which by integration, becomes:

$$V(S;t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_0^{\infty} e^{-\frac{\left[\log\left(\frac{S}{S^{'}}\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)^2\right]}{2\sigma^2(T-t)}} \cdot Payoff(S^{'}) \frac{dS^{'}}{S^{'}} \qquad (13)$$

where:
Payoff(S)=V(S;T) → ie: „profit/or loss for S
is equal to the option V, at time t=T" (14)

There are other analytical methods to assess Black-Scholes equation, for example: Fourier and Laplace transformations.

The methods numericede solving partial differential equations, parabolic type (that is, and the equation: Black-Scholes), we can mention:

Finite difference method with the following:

- explicit finite difference method and

- implicit finite difference method

Very popular in the literature (to category: „numerical methods") is the *Crank-Nicolson method,* which performs: „mean between the explicit and implicit method".

Explicit method, applied Black-Scholes equation, to implement Black-Scholes model is applied to options: call and put European-style (or American type).

Default method, Crank-Nicolson equation applied to options: call and put European type, is more difficult to program, but has greater precision.

Final conclusion:

Black-Scholes Equation $\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0$ represents a partial differential equation, parabolic type which corresponds to the Black-Scholes Model published by the American economists Fischer Black and Myron Scholes in 1973. This model may be used with success in evaluating equity options.

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