ECONOMIC PROBLEMS OF CONDITIONED OPTIMIZING

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A lot of economic problems are reduced to the optimization objective function, a multitude of admissible solutions.
Conditional optimization problems are grouped into two categories / types:

Optimization condition, restriction of equality;
Conditional optimization with inequality-type restrictions.

Next we will briefly deal: If a general and practical application for the two types of conditional optimization.

Keywords: *Objective function, conditioned optimizing, restriction of equality / inequality, Lagrange function, Hessian Matrix*

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1 **Optimization with conditional equality type restrictions**
The general case
Gives the functions:

$$f : R^n \to R; (x_1, x_2, ..., x_n) \mapsto f(x_1, x_2, ..., x_n) = f(x)$$
$$and$$
$$g_j : R^n \to R; (x_1, x_2, ..., x_n) \mapsto g_j(x_1, x_2, ..., x_n)$$

Required: optimization objective function f(x), namely: [Opt]f(x)=?; when there restrictions:

$$g_j(x_1, x_2, ..., x_m) = c_j; j = \overline{1, m}$$

Resolution: use the Lagrange method, which involves going through the following stages and understage / steps.
Stages and steps in solving:
$e_1) p_1)$ Economic problems associated Lagrange function, defined as:

$$L : R^n \times R^m \to R; (x; \lambda^*) \to L(x; \lambda) = f(x) + \sum_{j=1}^{m} \lambda_j \left[g_j(x) - c_j \right]$$

or developed:

$$L(x_1, x_2, ..., x_n; \lambda_1, \lambda_{2,}, ..., \lambda_m) = f(x_1, x_2, ..., x_n) + \sum_{j=1}^{m} \lambda_j \left[g_j(x) - c_j \right] \quad (1)$$

$p_2)$ Determine the stationary points as solutions of the equations, obtained by derivation of Lagrange function with respect to variables: $x_1, x_2, \ldots, x_n$ and of Lagrange multiplier $\lambda_j;\ j = \overline{1, m},$ namely:

$$(S)\begin{cases}\dfrac{\partial L}{\partial x_i}=0;\ i=\overline{1,n}\\ şi\\ \dfrac{\partial L}{\partial \lambda_j}=0;\ j=\overline{1,m}\end{cases}\Leftrightarrow\begin{cases}\dfrac{\partial L}{\partial x_1}=0\\ \dfrac{\partial L}{\partial x_2}=0\\ \vdots\\ \dfrac{\partial L}{\partial x_n}=0\end{cases}(+)\begin{cases}\dfrac{\partial L}{\partial \lambda_1}=0\\ \dfrac{\partial L}{\partial \lambda_2}=0\\ \vdots\\ \dfrac{\partial L}{\partial \lambda_m}=0\end{cases}\quad (S`) \tag{2}$$

p_3) Calculate the hessian matrix for the position: $\phi(x)=L(x;X^*)$, namely:

$$H_\phi(x)=\left(\frac{\partial^2\phi}{\partial x_i\cdot\partial x_j}\right)_{i=\overline{1,n};\quad j=\overline{1,m}} \tag{3}$$

p_4) It establishes the nature of symmetric matrix: $H_\phi(x^*)$

Phase II of / steps to resolve:

e_2) p'_1) Write form generated by the square matrix $H_\phi(x^*)$, namely:

$$d^2L=(dx)^T\cdot H_\phi(x^*)dx; \tag{4}$$

Note: the relationship (4), differential order is the second function of Lagrange: $L(x;\lambda)$.

p'_2) Differ restrictions of equality:

$$g_j(x_1,x_2,\dots,x_m)-c_j=0,\quad j=\overline{1,m} \tag{5}$$

p'_3) It establishes the nature of the square shape (reduced), which has only „n-r" variable.

Discussion:

Case I If the square shape is defined negatively, then the stationary point is a maximum.

Case II If the square is positive defined, then the stationary point is a minimum.

2. Practical application (the Lagrange method)

Determine $(*)\begin{cases}[Max]\ f(x)=?,\ dacă\ f(x)=\sqrt{x_1}\cdot x_2\\ 2x_1+x_2=5\ \ (system\ of\ restriction)\\ x_1>0;\ x_2>0\ (condition\ of\ positive)\end{cases}$

Resolution: we will use Lagrange following stages and steps to resolve, described previously, the general case.

Phase I / steps:

p_1) Lagrange function, the problem associated data (*), become:

$$L:R^2\times R\to R;\ (x_1,x_2,\lambda)\mapsto L(x_1,x_2,\lambda)=\sqrt{x_1}\cdot x_2+\lambda(2x_1+x_2-5) \tag{6}$$

or equivalent: $L(x_1,x_2)=x_1^{\frac{1}{2}}\cdot x_2+\lambda(2x_1+x_2-5)$

p_2) Determine stationary points as solutions of the system „3=2+1" equations with partial derivatives, obtained by derivation Lagrange function in relation to variables $x_{1,}x_2$ and the Lagrange multiplier λ, ie:

$$(S)\begin{cases}\frac{\partial L}{\partial x_i}=0\\ \frac{\partial L}{\partial x_2}=0\\ \frac{\partial L}{\partial \lambda}=0\end{cases}\Leftrightarrow\begin{cases}[\sqrt{x_1}\cdot x_2+\lambda(2x_1+x_2-5)]'_{x_1}=0\\ [\sqrt{x_1}\cdot x_2+\lambda(2x_1+x_2-5)]'_{x_2}=0\\ [\sqrt{x_1}\cdot x_2+\lambda(2x_1+x_2-5)]'_{\lambda}=0\end{cases}\Leftrightarrow \quad (7)$$

$$\Leftrightarrow (S')\begin{cases}\frac{x_2}{2\sqrt{x_1}}+2\lambda=0\\ \sqrt{x_1}+\lambda=0\\ 2x_1+x_2=5\end{cases} \quad (8)$$

Of the first two equations of system (S'), we obtain: $x_1=\frac{5}{6}$ $x_2=\frac{20}{6}$ (9) $\lambda=-\frac{\sqrt{30}}{6}$ (10)

p_3) Conclusion (part):

Stationary point $P(x_1^*,x_2^*,\lambda^*)$, with coordinates: x_1^*,x_2^*,λ^*, solutions of the system (S), becomes:

$$\begin{cases}x_1^*=\frac{5}{6}\\ x_2^*=\frac{20}{6}\\ \lambda^*=-\frac{\sqrt{30}}{6}\end{cases} \quad (11)$$

That it:

$$P(x_1^*,x_2^*,\lambda^*)=P(\frac{5}{6};\frac{20}{6};-\frac{\sqrt{30}}{6}) \quad (stationary\ po\text{int}) \quad (12)$$

p_4) Write function expression

$$\phi(x)=L(x;\lambda^*)=f(x)+\sum_{j=1}^{2}\lambda_j[g_j(x)-c_j] \quad (13)$$

where: $$\begin{cases}f(x)=\sqrt{x_1}\cdot x_2\\ \sum_{j=1}^{2}\lambda_j[g_j(x)-c_j]=-\frac{\sqrt{30}}{6}(2x_1+x_2-5)\end{cases} \quad (14)$$

Replace: (14) → (13) and obtain:

$$\phi(x)=\sqrt{x_1}\cdot x_2-\frac{\sqrt{30}}{6}(2x_1+x_2-5) \quad (15)$$

p_5) Hessiene write the matrix expression:

$$H_{\phi}(x) = \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j} \right)_{\substack{i=\overline{1,n} \\ j=\overline{1,m}}} \tag{16}$$

proper function $\phi(x) = L(x;\lambda^*)$

Note: The hessian matrix $\phi(x)$, in this case will take the form:

$$H_{\phi}(x) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{17}$$

p_6) Calculating actual matrix elements $H_{\phi}(x)$, we obtain:

$$A_{11} = \frac{\lambda^2 L}{\partial x_1^2} = \frac{\partial}{\partial x_1}\left(\frac{\partial L}{\partial x_1}\right) = \frac{\partial}{\partial x_1}\left(\frac{x_2}{2\sqrt{x_1}} + 2\lambda\right) = \left(\frac{x_2}{2\sqrt{x_1}} + 2\lambda\right)'_{x_1} = -\frac{x_2}{4x_1\sqrt{x_1}} \tag{18}$$

By similarly, for other items we have:

$$A_{22} = 0 \qquad A_{12}\frac{1}{2\sqrt{x_1}} \qquad A_{21} = \frac{1}{2\sqrt{x_1}} \tag{19}$$

p_7) Conclusion (part):

Hessian matrix $H_{\phi}(x)$, becomes:

$$H_{\phi} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} -\frac{x_2}{4x_1\sqrt{x_1}} & \frac{1}{2\sqrt{x_1}} \\ \frac{1}{2\sqrt{x_1}} & 0 \end{pmatrix} \tag{20}$$

p_8) Symmetric matrix $H_{\phi}(x)$ is obtained by substituting relations (20), previously calculated values for x_1 *şi* x_2, respectively: $x_1^* = \frac{5}{6}$ *şi* $x_2^* = \frac{20}{6}$;

That it:

$$A_{11}^* = -\frac{x_2^*}{4x_1^*\sqrt{x_1^*}} = -\frac{\frac{20}{6}}{4\frac{5}{6}\sqrt{\frac{5}{6}}} = -\frac{\sqrt{30}}{10};$$

Respectively:

$$A_{12}^* = \frac{\sqrt{30}}{10} \qquad A_{21}^* = \frac{1}{2\sqrt{x_1^*}} = \frac{\sqrt{30}}{10} \qquad A_{22}^* = 0 \tag{21}$$

Replacing the variables, data relations (28),in the symmetric matrix:

$$H_{\phi}(x^*) = \begin{pmatrix} A_{11}^* & A_{12}^* \\ A_{21}^* & A_{22}^* \end{pmatrix}$$

We have finally:

$$H_{\phi}(x^*) = \begin{pmatrix} -\frac{\sqrt{30}}{5} & \frac{\sqrt{30}}{10} \\ \frac{\sqrt{30}}{10} & 0 \end{pmatrix} \tag{22}$$

Note: The resulting expression matrix $H_{\phi}(x^*)$, under symmetric „theory” that:

$$A_{11}^{*} = -\frac{\sqrt{30}}{5} < 0,$$

that it:

The matrix $H_{\phi}(x)$ is of finite seminegativ (q.e.d). (23)

Phase II of steps to resolve:

$p_1^{'})$ Write square shape, generated by the matrix $H_{\phi}(x)$, which has the general form:

$$d^2 L = (dx)^T \cdot H_{\phi}(x^*) dx \tag{24}$$

where:

$$\begin{cases} dx = \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} \\ (dx)^T = (dx_1 \; dx_2) \\ H_{\phi}(x^*) = \begin{pmatrix} -\frac{\sqrt{30}}{5} & \frac{\sqrt{30}}{10} \\ \frac{\sqrt{30}}{10} & 0 \end{pmatrix} \end{cases} \tag{25}$$

$p_2^{'})$ Substituting relations (25) relationship (24), we obtain:

$$d^2 L = (dx_1 \; dx_2) \begin{pmatrix} -\frac{\sqrt{30}}{5} & \frac{\sqrt{30}}{10} \\ \frac{\sqrt{30}}{10} & 0 \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} =$$

$$= \left(-\frac{\sqrt{30}}{5} dx_1 + \frac{\sqrt{30}}{10} dx_2 \quad \frac{\sqrt{30}}{10} dx_1 + 0 \right) \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} =$$

$$= \left(-\frac{\sqrt{30}}{5} (dx_1)^2 + \frac{\sqrt{30}}{10} dx_2 dx_1 + \frac{\sqrt{30}}{10} (dx_1)^2 + 0 \quad -\frac{\sqrt{30}}{5} dx_1 dx_2 + \frac{\sqrt{30}}{10} (dx_2)^2 + \frac{\sqrt{30}}{10} dx_1 dx_2 + 0 \right) =$$

$$= \left(-\frac{\sqrt{30}}{10} dx_1^2 \quad \frac{\sqrt{30}}{10} dx_2^2 \right)$$

So: $d^2 L = \left(-\frac{\sqrt{30}}{10} dx_1^2 \quad \frac{\sqrt{30}}{10} dx_2^2 \right)$ (expression of the square shape) (26)

$p_3^{'})$ Differentiation restriction of equality $2x_1 + x_2 = 5$ and get:

$$d(2x_1 + x_2) = d5 \Leftrightarrow dx_1 = -\frac{1}{2} dx_2 \tag{27}$$

$p_4^{'})$ Replace (34) → (33) and obtain:

$$d^2 L = -\frac{3\sqrt{30}}{10} dx_2^2 < 0 \tag{28}$$

According to „theory”, stationary point $P\left(\frac{5}{6}; \frac{20}{6}; -\frac{\sqrt{30}}{6} \right)$ is a maximum, ie:

$$\exists \quad Max\left(\frac{5}{6}; \frac{20}{6}; -\frac{\sqrt{30}}{6} \right) \tag{29}$$

Phase III

Final conclusion: the coordinates of the maximum point of the problem data

$$(*)\begin{cases}[Max]\ f(x)=\max(\sqrt{x_1}\cdot x_2) \\ 2x_1+x_2=5 \quad (\textit{the system restriction}) \\ x_1>0;\ x_2>0 \quad (\textit{the positive conditions})\end{cases}$$

are:

$$\left(\frac{5}{6};\frac{20}{6};-\frac{\sqrt{30}}{6}\right) \quad \text{Q.E.D.} \qquad / \qquad (30)$$

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