# THE COMPUTATION OF SOME PRIVATE PENSIONS

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*Abstract: The aim of this study is the development of few models for the private pensions such as retirement pension, disability pension and the widow pension. We also provide a numerical application of the elaborated models.*

*Key words: Single net premium, computational functions, private pension, lifetime rent, disability pension, widow pension.*

There are multiple problems regarding the pension systems. Due to the difficulties existing on the social pensions system, new products have erected products that assure incomes equal to insured wages for disability occurrence (approximately 90%) and an income correlated to the contributions and performances reached by the private pension fund administrator, for full time retirement.

A few months ago, the mandatory private pension system (the so-called second pillar) was implemented. For the excepted persons of the second pillar and for those who desire another pension, the third pillar (the facultative one) is also an alternative. Contributors to a facultative pension system should realize incomes and pay contributions to the social public system. Of course, depending on the envisaged pension, there are some requirements to fulfill, such as the age of 52 years and 6 months (at the maximum) and the payment of 90 monthly contributions, if the goal is a lifetime pension, starting the age of 60, which represents the legal retirement one.

As for the facultative private pensions, the contributions are also added to the participant’s personal account. If he dies before he reached the legal retirement age, the cumulated amount shall be transferred to his legal heirs’ account. The contribution flexibility is also an advantage for the facultative private pensions, the insured person having the opportunity to raise the contribution (up to 15% of the taxable income) or to cease the payment of contributions the moment he does not realize incomes, but with the possibility to restart the contribution, once he will obtain them again. There is also an advantage the fact that the legal framework for private pensions is stable and the field of activity under continuous supervision.

The pension prediction still remains a complicate issue. Obviously, it can be an answer such as “the pension depends on contributions”. But this is totally vague and the hard work is switched to later computation (the moment all the data are known). There are a lot of variables to evaluate: commissions, taxes, penalties, investments and accompanying risks, transfer possibilities, payments of pensions. However, it can be obtained a perspective on cumulated amount from contributions by comparison to life insurances with accumulation paid as lifetime rents. We should also mention that the private pension represent a way of saving money for old age, meanwhile a life insurance with capital accumulation is mainly o protection product (mixed insurance) and secondly a saving opportunity. As a consequence, a pension plan means contracting annuities or postponed lifetime fractionalises or certain benefits according to the contributions on the active life.

The most simple mathematic method for full time pension is the one that enables the insured person, x years old, to pay for the following r years, $m$ times a year, a net premium $P$, in order to obtain from the insurer, over n years ($r \le n$), a $k$ times a year a pension $S$ till $t$ years. We note $a_x^{(b,c)}(\alpha:\beta)$ the actual medium value of a series of fractionalises (which represent an annually monetary unit) made for the period $[\alpha,\beta]$, where $b$ indicates the anticipated or posticipated payment ($A$, $P$) and $c$ the annual fraction. As a result we obtain the equation: $m\cdot P\cdot a_x^{(b,m)}(:r)=k\cdot S\cdot a_x^{(d,k)}(n:t)$, where $b,d\in\{A,P\}$.

For the disability insurance, we shall consider three possible states: the „a” state means active (apt or valid) for a named job, the „i” means disable for every job and the „d” state (deceased). We presume the decease as the only way to eliminate the “i” state. We note with : $l_x^i$ the number of living individuals in “i” state, at age x; $l_x^a$ the number of living individuals in "a" state; $l_x^{ai}$ the number of living individuals changing the „*a*" state in „*i*" state at age $y \in (a, x)$ and remaining in this state until the age of $x$; $p^{ii}(y, x)$ the probability that the individual in state "i", at age y will be in the same state at the age x; $p^{ai}(y, x)$ the probability that the individual in state "d" at age y will pass and remain in state "i" until age x. We note with $l_x^i$ the surviving function for disabled persons and with $q_x^i$ the decease annual rate. We will use the commutation functions (numbers):

$$D_x^i = l_x^i \cdot v^x ,\ N_x^i = \sum_{y=x}^{\omega} D_y^i ,\ C_x^i = \left(l_x^i - l_{x+1}^i\right) \cdot v^{x+\frac{1}{2}} ,\ M_x^i = \sum_{y=x}^{\omega} C_y^i .$$

For an "n" years survival insurance for a disabled person *x* years old, the single net premium for an insured amount of one u.m. is: $E_{x,n}^i = p^{ii}\left(x, x+n\right) \cdot v^n = \frac{l_{x+n}^i}{l_x^i} \cdot v^n = \frac{D_{x+n}^i}{D_x^i}$.

For one u.m. rent, postponed *r* years and limited to *n* years, paid annually and anticipated to a disabled person of *x* years old for the time he is alive, the single net premium is: $a_x^{i,(A)}\left(r:n\right) = \frac{N_{x+r}^i - N_{x+n}^i}{D_x^i}$.

For the anticipated life annuity, arithmetical progression of rate 1 *u.m.* and first term 1 u.m., postponed *r* years and limited to *n* years, the single net premium is:

$$Ia_x^{i,(A)}\left(r:n\right) = \frac{S_{x+r}^i - S_{x+n}^i - \left(n-r\right) \cdot N_{x+n}^i}{D_x^i}$$

We shall determine now the single net premium paid by an active individual, *x* years old, for an insurance that offers him as soon he is getting disable, an annually anticipated life rent, with the annuities of 1 u.m., immediately and unlimited, but only in case the event occur between *x* + *r* and *x* + *n* years. We shall note this premium to $a_x^{ai,(A)}\left(r:n\right)$. Later, we shall obtain the single net premium paid by the *x* years old active person, in order to receive, immediately and unlimited, an annually anticipated life rent, but only in case the disability occur between *x* + *k* and *x* + *k* + 1 years. We shall note this premium $E_{x,k}^{ai}$. Presuming the disability cases uniform distributed through the year, we may conclude the moment an active person become disable to be the middle of the year. In this case, the value for the disability pension at that moment is $a_{x+k+\frac{1}{2}}^{i,(A)}$. We have: $E_{x,k}^{ai} = \frac{l^{ai}\left(x+k, x+k+1\right)}{l_x^{aa}} \cdot \frac{v^{x+k+\frac{1}{2}}}{v^x} \cdot a_{x+k+\frac{1}{2}}^{i,(A)}$.

Introducing the commutation functions: $D_x^{aa} = l_x^{aa} \cdot v^x$, $N_x^{aa} = \sum_{y=x}^{\omega} D_y^{aa}$, $S_x^{aa} = \sum_{y=x}^{\omega} N_y^{aa}$, $D_x^{ai} = l^{ai}\left(x, x+1\right) \cdot a_{x+\frac{1}{2}}^{i,(A)} \cdot v^{x+\frac{1}{2}}$,

$N_x^{ai} = \sum_{y=x}^{\omega} D_y^{ai}$ and $S_x^{ai} = \sum_{y=x}^{\omega} N_y^{ai}$ we obtain: $E_{x,k}^{ai} = \frac{D_{x+k}^{ai}}{D_x^{aa}}$ and $a_x^{ai,(A)}\left(r:n\right) = \frac{N_{x+r}^{ai} - N_{x+n}^{ai}}{D_x^{aa}}$.

Let's consider the insurance by which a valid person, x years old, is going to receive between *x* + *r* and *x* + *n* years, as long as he is an active person, a life time anticipated annuity of 1 u.m. Noting with $a_x^{aa,(A)}\left(r:n\right)$ the single net premium for this insurance, we have:

$$a_x^{aa,(A)}\left(r:n\right) = \sum_{k=r}^{n-1} \frac{l_{x+k}^{aa}}{l_x^{aa}} \cdot v^k = \sum_{k=r}^{n-1} \frac{D_{x+k}^{aa}}{D_x^{aa}} = \frac{N_{x+r}^{aa} - N_{x+n}^{aa}}{D_x^{aa}}$$

Generally speaking, for the disability insurance we introduce a lack term and the progressive variation of the life time rent in consideration with the number of contributing years. We shall forward consider the case when the rent increases unlimited, in an arithmetical progression. This means that for a lack term of k years, the active insured person, x years old, recently contractor of this insurance, in case of disability between $x + k + 1,\ x + k + 2, \ldots$ shall receive a disability pension in arithmetical progression of step β and first termα.

We presume that the insured contracted the insurance *t* years ago. We note the medium actual value for this insurance with $AIN_{x,k,t}$ , for *t* = 0, this being similar to the single net premium for the insurance.

**1.** In case 0 ≤ *t* ≤ *k*, we have:

$$AIN_{x,k,t} = \sum_{j=0}^{\omega-x-k+t-1} (\alpha + j \cdot \beta) \cdot E_{x,k-t+j}^{ai} = \frac{\alpha \cdot N_{x+k-t}^{ai} + \beta \cdot S_{x+k-t+1}^{ai}}{D_x^{aa}}$$

The net single premium for the insurance is: $AIN_{x,k} = \dfrac{\alpha \cdot N_{x+k}^{ai} + \beta \cdot S_{x+k+1}^{ai}}{D_x^{aa}}$.

**2.** In case *t* > *k*, we have:

$$AIN_{x,k,t} = \sum_{j=0}^{\omega-x-1} \left[\alpha + (t-k) \cdot \beta + j \cdot \beta\right] \cdot E_{x,j}^{ai} = \left[\alpha + (t-k) \cdot \beta\right] \cdot \frac{N_x^{ai}}{D_x^{aa}} + \beta \cdot \frac{S_{x+1}^{ai}}{D_x^{aa}}$$

We consider now that the growth of the rent is limited to a specified number of years, and then this is set constant to the attained level. In this case, the new insured person shall receive after the lack period of k years in case of disability, a pension with the following anticipated annual values: $\alpha$ , $\alpha + \beta$ , $\alpha + 2 \cdot \beta$ , ... , $\alpha + n \cdot \beta$ , ...

We use the medium actual value with $AIL_{x,k,n,t}$ .

**1.** In case 0 ≤ *t* ≤ *k*, we have:

$$AIL_{x,k,n,t} = \alpha \cdot \frac{N_{x+k-t}^{ai}}{D_x^{aa}} + \beta \cdot \frac{S_{x+k-t+1}^{ai} - S_{x+k-t+n+1}^{ai}}{D_x^{aa}}.$$

The single net premium is: $AIL_{x,k,n} = \dfrac{\alpha \cdot N_{x+k}^{ai} + \beta \cdot \left(S_{x+k+1}^{ai} - S_{x+k+n+1}^{ai}\right)}{D_x^{aa}}$.

The moment the insurance and the lack term are the same, *t* = *k*, we have: $AIL_{x,k,n} = \dfrac{\alpha \cdot N_x^{ai} + \beta \cdot \left(S_{x+1}^{ai} - S_{x+n+1}^{ai}\right)}{D_x^{aa}}$.

**2.** In case *k* < *t* ≤ *k* + *n*, we have:

$$AIL_{x,k,n,t} = \frac{\alpha \cdot N_x^{ai} + \beta \cdot \left[S_{x+1}^{ai} - S_{x+k+n-t+1}^{ai} + (t-k) \cdot N_x^{ai}\right]}{D_x^{aa}}.$$

**3.** In case *t* > *k* + *n*, we have:

$$AIL_{x,k,n,t} = \sum_{j=0}^{\omega-x-1} (\alpha + n \cdot \beta) \cdot E_{x,j}^{ai} = (\alpha + n \cdot \beta) \cdot \frac{N_x^{ai}}{D_x^{aa}}.$$

For the widow pension, we note $a_y^w$ the medium actual value of an annuity paid to a widow, $y$ years old and $L_{\frac{x}{y}}$ the number of married men, $x$ years old married to wives, $y$ years old and $L_x$ the number of individuals, $x$ years old. Consequently, it results that the medium actual obligation of an insurer, due to the active person's decease, $x$ years old, is: $W_x = \frac{\sum_y L_{\frac{x}{y}} \cdot a_y^w}{L_x}$.

If $I_x$ is the number of active person, becoming disabled between $x$ and $x + 1$ years old, then we shall note $a_x^{iw}$ to be the medium actual obligation of an insurer to provide a pension for the disable's widow (the deceased was $x$ years old). Considering the deceases to be uniformly distributed during the year, we shall obtain: $a_x^{iw} = \sum_{k=0}^{\omega-x} \frac{d_{x+k}^i}{l_x^i} \cdot v^{k+\frac{1}{2}} \cdot W_{x+k+\frac{1}{2}}$.

Making use of the commutation functions: $D_x^{iw} = d_x^i \cdot v^{x+\frac{1}{2}} \cdot W_{x+\frac{1}{2}}$ and $N_x^{iw} = \sum_{k=0}^{\omega-x} D_{x+k}^{iw}$, we have: $a_x^{iw} = \frac{N_x^{iw}}{D_x^i}$. Noting $a_x^{aw}$ as actual medium value for the widow's insurance for an active person, $x$ years old, we have:

$$a_x^{aw} = \sum_{k=0}^{\omega-x} \left( \frac{d_{x+k}^{aa}}{l_x^{aa}} \cdot v^{k+\frac{1}{2}} \cdot W_{x+k+\frac{1}{2}} + \frac{I_{x+k}}{l_x^{aa}} \cdot v^{k+\frac{1}{2}} \cdot a_{x+k+\frac{1}{2}}^{iw} \right).$$

Introducing the commutation numbers: $D_x^{aw} = v^{x+\frac{1}{2}} \cdot \left( d_x^{aa} \cdot W_{x+\frac{1}{2}} + I_{x+k} \cdot a_{x+\frac{1}{2}}^{iw} \right)$ and $N_x^{aw} = \sum_{k=0}^{\omega-x} D_{x+k}^{aw}$, we obtain: $a_x^{aw} = \frac{N_x^{aw}}{D_x^{aa}}$.

In order to illustrate the previous elaborated models, we shall present an example. We shall consider an active husband, 32 years old, concluding an insurance contract, as it follows:

a) if he becomes disable in 10 years, he will receive a disability pension equal to 40% of his wage, pension that will increase for every work year by 3% of the wage (in 30 years is equal to the wage);
b) Beginning 62 years old, he will receive a full time pension, equal to the wage;
c) If his wife dies, he will receive a pension equal to 50% of his wage, unless his working period was less than 10 years. For these rights he is paying an annually contribution, equal to a percent $p$ of his wage, as long he is healthy till the age of 62. We shall determine the $p$ percent of the insured's contributions considering his wage to be constant for the rest of his working life. We shall note with $S$ the insured's wage and with $x$ ($x$ = 32 years) his age. The insurer's actual obligation is compound of:

− The actual value for the disability pension:

$$V_1 = \frac{0,4 \cdot S \cdot N_{42}^{ai} + 0,03 \cdot S \cdot \left( S_{43}^{ai} - S_{63}^{ai} \right)}{D_{32}^{aa}} = 0,74905 \cdot S.$$

If we cease the disability pension, once he will obtain the full time pension, then:

$$V_1 = \alpha \cdot \frac{N_{x+k}^{ai} - N_{x+k+n}^{ai}}{D_x^{aa}} + \beta \cdot \frac{S_{x+k+1}^{ai} - S_{x+k+n}^{ai} - (n-1) \cdot N_{x+k+n}^{ai}}{D_x^{aa}},$$

Where $\alpha = 0.4 \cdot S$, $\beta = 0.03 \cdot S$, $k = 10$ years the lack term and $n = 20$

We obtain:

$$V_1 = \frac{0,4 \cdot \left(N_{42}^{ai} - N_{62}^{ai}\right) + 0,03 \cdot \left(S_{43}^{ai} - S_{62}^{ai} - 19 \cdot N_{62}^{ai}\right)}{D_{32}^{aa}} \cdot S = 0.40746 \cdot S.$$

- The actual value for the full time pension: $V_2 = S \cdot \frac{N_{62}^{aa}}{D_{32}^{aa}} = 0{,}591715 \cdot S$ u.m.
- The actual value for the widow pension: $V_3 = 0,5 \cdot S \cdot \frac{N_{42}^{aw}}{D_{32}^{aa}} = 0{,}23013 \cdot S.$

The actual value for the insured's contribution is:

$$V = p \cdot S \cdot \frac{N_{32}^{aa} - N_{62}^{aa}}{D_{32}^{aa}} = 13{,}789864 \cdot p \cdot S \quad \text{u.m.}$$

So, in case of addition of full time pension and disability pension, we have:

$$p = \frac{V_1 + V_2 + V_3}{13,789864 \cdot S} = \frac{0,74905 + 0,591715 + 0,23013}{13,789864} = 0,114 = 11,4\%.$$

If the payment of disability pension is ceased, the moment the person receives full time pension, we shall have:

$$p = \frac{V_1 + V_2 + V_3}{13,789864 \cdot S} = \frac{0,40746 + 0,591715 + 0,23013}{13,789864} = 0,089 = 8,9\%$$

Obviously, the high percent of contribution is determined by the level of important requested benefits. In case of separate treatment of the two previous pensions, the contributions decrease at least two percent.

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