CHAOS MODELS IN ECONOMICS

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Abstract – The paper discusses the main ideas of the chaos theory and presents mainly the importance of the nonlinearities in the mathematical models. Chaos and order are apparently two opposite terms. The fact that in chaos can be found a certain precise symmetry (Feigenbaum numbers) is even more surprising. As an illustration of the ubiquity of chaos, two models among many other existing models that have chaotic features are presented here: the nonlinear feedback profit model and one model for the simulation of the exchange rate.

Keywords: chaos, nonlinear systems, complex behavior, bifurcation diagram

Introduction

One of the axioms of the modern science asserts that if an accurate description of a physical system can be identified then the possibility of a deeper understanding of the system and the prediction of the system evolution is possible.

These assertions are not always correct. For instance, if one applies the laws of motion stated by Newton, then there is possible to predict exactly the orbit of the Moon around the Earth if the influence of other planets is not considered. These predictions were verified and proved to be accurate. If the third planet is included, the mathematical model of the interaction of the two bodies becomes "the three bodies problem", solved by Newton but for a limited set of cases and unsolved for the general case. Today by means of a computer, "the tree bodies problem" can be solved, but one can observe that the prediction of the orbit of the third planet is often impossible.

A large number of real systems have a nonlinear behavior despite the idealized linear behavior used in modeling. The development of a new way of dealing with nonlinear systems is obvious. This "new way of dealing" exists already despite the fact that the study of the nonlinearity is still at the beginning.

Some changes in nonlinear systems can lead to a complex and erratic behavior called chaos. The nonlinearity is one of the conditions needed by a system in order to develop chaos. The term chaos is used to describe the behavior of a system that is aperiodic and apparently random.

S. H. Strogatz defines chaos as an aperiodic long time behavior developed by a deterministic system highly sensitive on initial condition. [1] Behind this apparently random behavior lies the deterministic character determined by the equations describing the system. Most of the systems that are used as examples to explain the concepts of chaos theory are deterministic.

There are two types of chaos: deterministic and nondeterministic. The deterministic chaos represents the chaotic motion of the nonlinear systems whose dynamic laws determines uniquely the evolution of the system's state based on the previous evolution.

The deterministic chaos represents only one particular case of what is called nondeterministic chaos that exhibits a superexponential divergence of the trajectories. In this case the equations describing the evolution of the system are not known. The both ways of chaos manifestations are short-term predictable but long term unpredictable.

The chaos and the concepts related to the dynamics of the systems and the their modeling using differential equations is named the chaos theory and is tightly related with the notion of nonlinearity [4]. The nonlinearity implies the loss of the causality correlation between the perturbation and effect propagated in time. The study of the nonlinearity is named nonlinear dynamics – a captivating domain using a mathematical apparatus still under development.

Despite the fact that the ideas leading to the emergence of the chaos theory existed before longtime, Lorenz (1963) created a mathematical model of the convection currents circulation in atmosphere and observed that when the systems begins with initial conditions slightly changed from the previous ones, the results are completely different. This phenomenon will lie at the basis of a very popular paradigm of chaos named "the butterfly effect", that states that if the flapping of a butterfly slightly modifies the atmospherically conditions in the Amazonian jungle, this fact can have an impact, at the end of a complex cause – effect chain in setting off a tornado in Texas.

The butterfly effect paradigm contains the essence of the phenomenon characterizing the chaos: first, the sensitive dependence on initial conditions and second – the fact that to predict the future state of a chaotic system, the current state need to be known with infinite prediction.

The manifestation of chaos can be found everywhere in the real world, for instance: the propagation of the avalanches, epidemics spreading, climate evolution, heart beats, lasers, electronic circuits, etc.

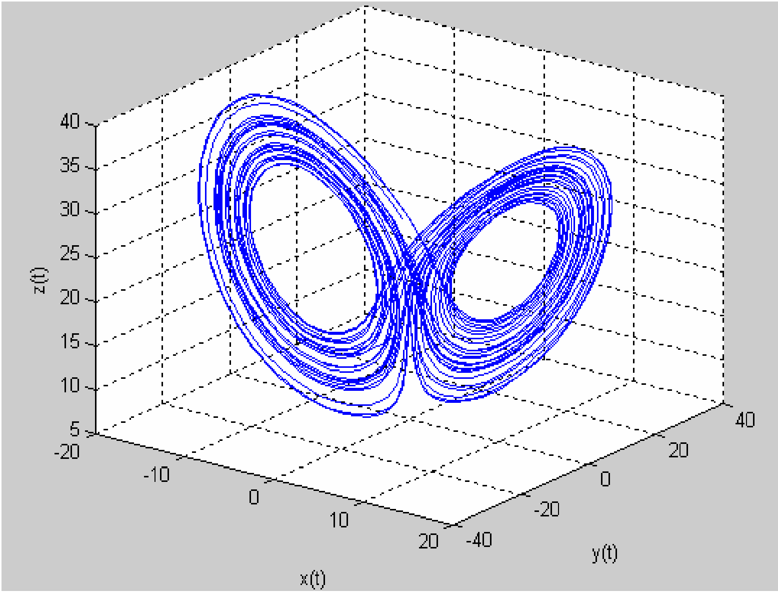


Figure 1. Lorentz attractor – the "butterfly" of chaos theory.

A legitimate question is that the chaos is the rule or the exception from the rule. Taking into account that most of the systems of the real world are nonlinear (the basic condition for the emergence of chaos), seems that chaos could be one of the not so obvious features of the nature.

The importance of studying chaos is that chaos offers an alternate method that explains the apparently random behavior of the complex systems. The chaos plus the specific mathematical tools is a framework of studying different models from different fields, models that can be reduced to elementary models with known chaotic behavior for some values of the parameters.

The way to chaos begins with the phenomenon of period doubling. The period doubling evolves in 2, 4, 8, 16 and so on periods and the system evolution can abruptly fall into chaotic regime.

In the case of unimodal function there is an interesting symmetry in the parameter values for what the period doubling occurs.

If A_1 is the value of the control parameter for what the first period doubling occurs and A_n is the value for what the n^{th} period doubling occurs, then:

$$\delta = \lim_{n \to \infty} \frac{A_n - A_{n-1}}{A_{n+1} - A_n} = 4.66920 \qquad (1)$$

where δ is the Feigenbaum number valable for all unimodal functions.[5]

Nonlinear Models

A. Chaos in exchange rates

For the simulation of the volatile behavior of the exchange rates were created models that treat the exchange rates as being prices of the financial assessments traded on efficient markets. The current exchange rate contains the currently available information and the changes observed reflect the effect of the new events that are unpredictable by definition.

The theory states that an accurate a priori prediction of the exchange rate evolution is impossible to be made but the subsequent explanation of the changes is possible. In order to eliminate these difficulties, the chaos theory and the nonlinear models are extensively used. The first researches have been carried out starting from 1980.

In the majority of situations these models are highly nonlinear and result in a wide range of dynamic behavior, including chaotic dynamics. There is a dispute over the manifestation of chaotic dynamics in exchange rates. There are many studies that are positive to the chaotic dynamics (Federici 2001, Westerhoff, Darvas 1998, Hommes 2005, Vandrocicz 2006) and also a number of studies that are rejecting the chaos in exchange rate (Brooks, Serletis).

The chaos theory demonstrates that even the simplest dynamical systems can exhibit at some point a very complex behavior. If the exchange rates variation is caused due to the chaotic nature of the system, this should lead to the fact that the smallest influences should have the effect of a nonlinearity over the exchange rates – exactly what happens in reality.

The first model presented demonstrates the fact that even the simplest models can exhibit chaotic behavior. [3]

The demand of foreign currency is determined as percentage of the deviation of current exchange rate towards the expected one.[2]

$$S_t = \alpha\left(\frac{e^e}{e_t} - 1\right), \alpha \geq 0 \tag{2}$$

where

e_t is the domestic price of the foreign currency

e^e is the future estimated exchange rate

α is the sensitivity parameter

The trade balance (T_i) is a linear function depending on the current exchange rates and the corresponding exchange rate for the last period, written as deviation from the expected values and is given by the equation:

The expected exchange rate represents the stable state at which the speculators on the market do not wish to sell nor buy.

$$T_t = \beta\left(e_t - e^e\right) + \gamma\left(e_{t-1} - e^e\right) \ \beta, \gamma > 0 \tag{3}$$

The clearing of the exchange markets writes as:

$$\Delta S_t = T_t \tag{4}$$

After replacing equations (2) and (1) in (4), equation (4) becomes:

$$\beta e_{t-1} e_t^2 - \left[(\beta + \gamma) e^* e_{t-1} - \gamma e_{t-1}^2 - a e^*\right] e_t - \alpha e^* e_{t-1} = 0 \tag{5}$$

The equation 5 has two roots, the positive one being considered for obvious reasons. The resulting nonlinear equation is:

$$e_t = \frac{\left[(\beta+\gamma)e^* e_{t-1} - \gamma e_{t-1}^2 - ae^*\right]}{2\beta e_{t-1}} + \frac{\sqrt{\left[(\beta+\gamma)e^* e_{t-1} - \gamma e_{t-1}^2 - ae^*\right]^2 + 4 * \beta e_{t-1} * \alpha * e_{t-1}}}{2\beta e_{t-1}} \tag{6}$$

for α=β=4 and γ=26.

The graphical representation of the solution e_t show that the graph presents a peak value of 2.76 and a minimum value of 0.091. Any other value from outside the interval represented by these two values is attracted. The evolution of the system with the specified parameters is chaotic because satisfies the Ly-Yorke condition [3].

The Figure 2 illustrates the evolution of the system for two initial slightly different values: 0.2 and 0.2005 (the dotted line). The values of the two time series are identical for a short period of time (the first 10 iterations) and then the trajectories of the systems are diverging.

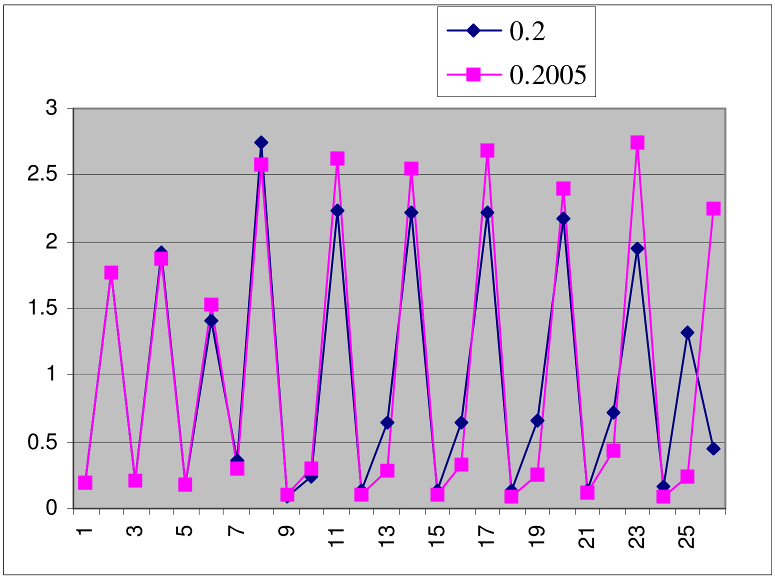
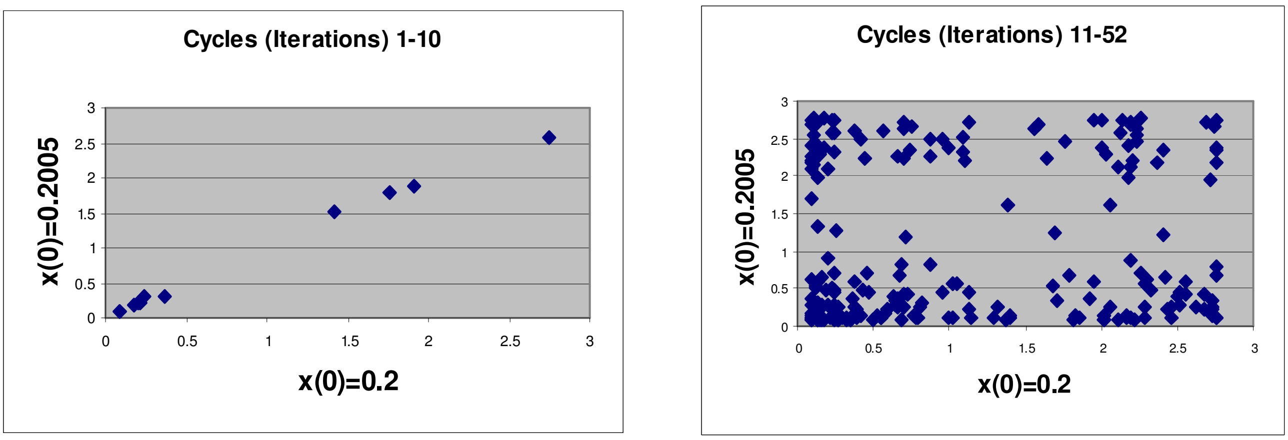


Figure 2. The influence of the initial conditions.

The scatterplots for the two time series are provided to demonstrate the independence of the two time series after 10 iterations. The scatterplots presented in Figure 3 and Figure 4 one of the fingerprints of chaos: the distance between two trajectories starting from nearby points in the state space diverge over time.



a b

Figure 3a. The scatterplot for the first 10 iterations and b) the scatterplot for the last 41 iterations.

When the sensitivity parameter is varied, the same effects can be observed. Figure 4 presents the trajectories of the system for two very near values of α.

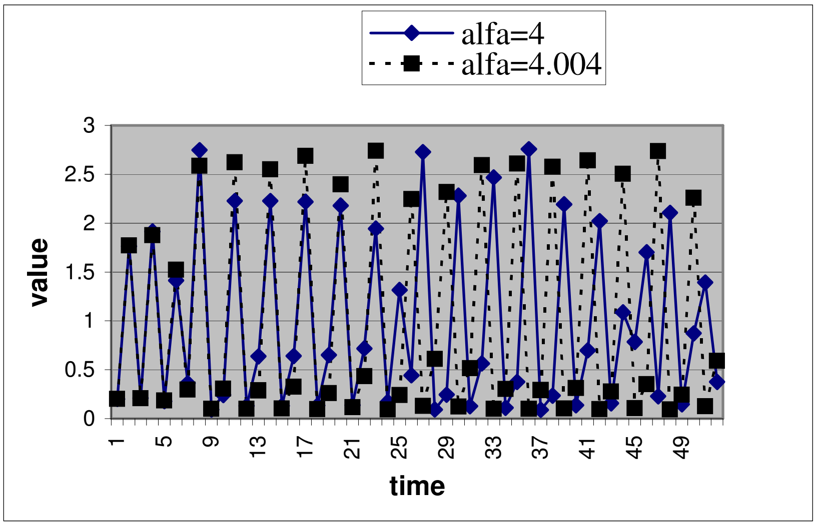


Figure 4. The influence of changing sensitivity parameter.

The apparently irrelevant changes can affect the longtime behavior of the exchange rate modeled using the Ellis model and some of these small shocks can determine the system to fall into the chaotic regime.

B. The model of the nonlinear feedback mechanism of the profit

The current spending of a firm can influence the value of the profit obtained at the end of the reference period. The profit will influence the spending over the next period. The dependence between the previous value of the profit and the current value is nonlinear because an increase of the spending does not reflect in an increase of the profit. The law of the decrease of the efficaciousness asserts that a certain mean value reaches minimum or maximum value when its magnitude equals the marginal value. One can invest in a certain production capability but this doesn't guarantee an unlimited increase of the production but the increase up to a certain point. Beyond that point the increase of the investment does not generates a corresponding increase of the production.

The dependence between the current profit and the previous profit can be modeled by using the equation:

$$\Pi_{t+1} = A\Pi_t - B\Pi_t^2 \tag{6}$$

The maximum profit Π_{max} .is supposed that it can be determined.

Dividing the equation (6) with Π_{max} the following result is obtained:

$$\frac{\Pi_{t+1}}{\Pi_{max}} = A\frac{\Pi_t}{\Pi^{max}} - B\left(\frac{\Pi_t}{\Pi^{max}}\right)^2 \Pi^{max} \tag{7}$$

Let $\pi_t = \dfrac{\Pi_t}{\Pi^{max}}$ and the equation (7) becomes:

$$\pi_{t+1} = A\pi_t - B\pi_t\Pi^{max} \tag{8}$$

If we take $\Pi^{max} = \dfrac{A}{B}$ the equation above becomes the logistic equation:

$$\pi_{t+1} = A\pi_t - A\pi_t^2 = A(1-\pi_t)\pi_t \tag{9}$$

The logistic map exhibits the same dependence on the initial condition: the slightest change of the initial condition causes a completely different evolution.

The complex behavior of the apparently simple functions can be observed using the bifurcation diagram. The bifurcation diagram (Figure 5) is an excellent tool allowing analyzing the behavior of a function by varying a control parameter (in the case of logistic function, the control parameter is A).

The logistic function is known to have a chaotic behavior with small isles of periodicity for a value of the parameter A greater that 3.57. For A∈ [3.57, 4] there are small areas of periodicity, the white stripes that can be observed in the figure. For A>4 the behavior is completely chaotic.

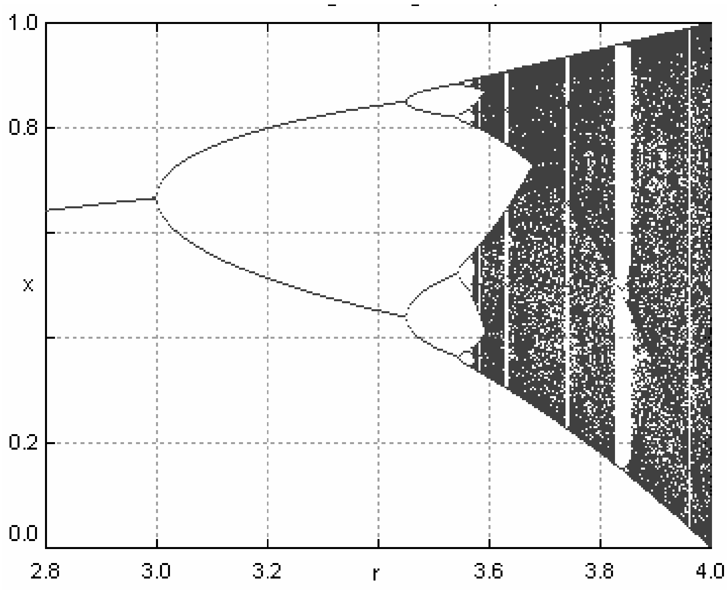


Figure 5. The bifurcation diagram for the logistic function

Conclusions

Chaos is can be found almost everywhere in the nature. Chaos theory and fractals are currently applied in the study of the natural phenomenon.

An essential condition needed in order that chaos to emerge is to have nonlinear systems. In fact very few of all models are purely linear, the vast majority of the systems are nonlinear.

The paper emphasizes two of the features of the chaotic systems: dependence to initial conditions and the divergence of nearby trajectories.

Two of the models used in economy that could exhibit chaos are described and discussed.

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