ABOUT RISK PROCESS ESTIMATION TECHNIQUES EMPLOYED BY A VIRTUAL ORGANIZATION WHICH IS DIRECTED TOWARDS THE INSURANCE BUSINESS

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Abstract. *In a virtual organization directed on the insurance business, the estimations of the risk process and of the ruin probability are important concerns: for researchers, at the theoretical level, and for the management of the company, as these influence the insurer strategy. We consider the evolution over an extended period of time of the insurer surplus process. In this paper, we present some methods for the estimation of the ruin probability and for the evaluation of a reserve fund. We discuss the ruin probability with respect to: the parameters of the individual claim distribution, the load factor of premiums and the intensity parameter of the number of claims process. We analyze the model in which the premiums are computed according to the mean value principle. Also, we attempt the case when the initial capital is proportional to the expected value of the individual claim. We give numerical illustration.*

Key Words: virtual organization, ruin probability, risk process, adjustment coefficient.

1. Introduction

When an insurance company issues certain products, it is useful for such a company to build up a virtual organization which is meant to conceive, sell and manage in a better way (manner) those products. This fact follows from the advantages brought by the flexibility and mobility of a virtual organization, as well as from the possibilities to optimize costs and risks implied by such an organization.

The problems related to the payment insolvency and the ruin of an economic agent (insurance company, insurer) are of particular interest for the researchers concerned with the economic phenomena. The economic and mathematical modeling of ruin has generated many works, the latest ones being those worked out by Soren Asmussen [1], Dickson and Willmot [4], Garrido and Li [6], Stanford and his assistants [2] and Zbaganu [11].

In these papers, we have noticed the behavior of the ruin probabilities as a function of the initial capital of an insurance virtual company and to the load factor which used to settle the tariff premiums. Our conclusion is that for the classical risk process (where the process of the number of damage claims is a homogeneous Poisson process), the use of the mean value principle to compute the net premiums gives a much too powerful dependency between the cash-flows of the input and output system (cashes and payments of the company). At the same time, we have approached the estimation of the ruin probability in the case when the moments generating function of the random variables describing the individual claims does not exist. Also, we have dealt with the evaluation of the minimum reserve of risk for certain accepted levels of the ruin probability.

2. Theoretical foundations

The risk model

We shall denote by:

$C(t)$ the capital (or the cash-flow) of the company at moment t;

r – the initial capital, hence $r = C(0)$;

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$D(t)$ – the total damage paid by the insurance company till moment t (in short, the total claim);

X_i – the i-th individual claim;

$N(t)$ – the number of individual claims up to moment t, so $D(t) = \sum_{i=1}^{N(t)} X_i$;

c – the net average income per time unit;

θ - the load factor of the premiums.

We can write $C(t) = r + c \cdot t - D(t)$.

We shall consider that: the stochastic process $\{N(t)\}_t$ is a homogeneous Poisson process of parameter λ, the individual claims are independent random variables (independent also of $N(t)$) and identically distributed (i.i.d.), and that we use the mean value principle in order to compute the net premiums, thus: $c = (1+\theta) \cdot \lambda \cdot m_1 = g(\theta)$, where m_1 is the expected value of the individual claim ($m_1 = \mathbf{E}X$). We shall define the ruin for the mathematical model, as the situation when the company capital takes a negative value, and we shall denote by τ the ruin moment, consequently: $\tau = \inf\{t \,|\, C(t) < 0\}$.

We denote by $\Psi_n(r,\theta)$, or $\Psi_n(r,\theta,m_1)$, the ruin probability till moment n, namely $\Psi_n(r,\theta) = P(\tau < n \,|\, C(0) = r, g(\theta) = c)$, and by $\Psi(r,\theta)$, or $\Psi(r,\theta,m_1)$, the ruin probability on an infinite time horizon, namely $\Psi(r,\theta) = P(\tau < \infty \,|\, C(0) = r, g(\theta) = c)$.

We give $\Psi(r,\theta) = \lim_{n \to \infty} \Psi_n(r,\theta)$.

The parameters r and θ, i.e. the initial reserve and the load factor of premiums, are deterministic and represent the instruments by which the insurance company (or the actuary) can act to diminish the ruin probability, in other words, to avoid the unpleasant event of ruin. The parameter m_1 is useful for various analyses. Denoting by $F = P \circ X^{-1}$ the cumulative distribution function of the individual claim, we get that $P(\tau \leq n \,|\, X_1) = \Psi_{n-1}(r - X_1, \theta)$,

$P(\tau < \infty \,|\, X_1) = \Psi(r - X_1, \theta), \Psi_n(r,\theta) = \mathbf{E}\Psi_{n-1}(r - X_1, \theta)$ and $\Psi(r,\theta) = \mathbf{E}\Psi(r - X_1\theta)$.

We denote by $S(\tau) = (-C(\tau) \,|\, \tau < \infty))$ the severity of the ruin at the moment of ruin, and by R the adjustment coefficient, namely the strictly positive solution of the equation: $\lambda + g(\theta) \cdot R = \lambda \cdot M_x(R)$, where M_x is the moment generating function of the individual claim (with $\omega(z) = \lambda(M_x(z) - 1) - g(\theta) \cdot z$).

The following result is obtained (Mircea, 2006: p. 210).

Proposition 1. When the adjustment coefficient R exists, we have:

i) The stochastic process $\left\{e^{-z \cdot C(t) - \omega(z)t}\right\}_t$ is a martingale for any $z \in \mathbf{R}$, with $M_x(z) < \infty$.

ii) The ruin probability is $\Psi(r,\theta) = \dfrac{e^{-R \cdot r}}{\mathbf{E}\left[e^{R \cdot S(\tau)}\right]}$.

In the case when the individual claims follow an exponential distribution, we obtain:

Consequence: If $X \sim Exp(\alpha)$,

then $R = \alpha - \lambda \cdot h(\theta)$ and $\Psi(r,\theta,\alpha,\lambda) = \dfrac{\lambda}{\alpha} \cdot h(\theta) \cdot e^{-(\alpha - \lambda \cdot h(\theta)) \cdot r}$, where $h(\theta) = \dfrac{1}{g(\theta)}$.

As the equation defining the adjustment coefficient is not easy to solve, we can find out the covering interval for it, an interval with the upper bound given by $R_s = \dfrac{2 \cdot \theta \cdot m_1}{m_2} > R.$ The lower bound is

$R > \dfrac{1}{m} \ln \dfrac{g(\theta)}{\lambda \cdot m_1}.$ assuming that the individual claim random variable is bounded by the constant real number m.

Definition. A distribution function F with $F(0)=0$ is called sub-exponential if $\lim\limits_{t \to \infty} \dfrac{1 - F^{*2}(t)}{1 - F(t)} = 2$.

Proposition 2. For $z > 0$, if F is sub-exponential, then $\lim\limits_{t \to \infty} e^{z \cdot t} \cdot (1 - F(t)) = \infty.$

Proof. For $0 < x < t$, we have: $e^{zt}(1 - F(t)) = \dfrac{1 - F(t)}{1 - F(t-x)} \cdot (1 - F(t-x)) \cdot e^{z \cdot (t-x)} \cdot e^{zx}.$

Let $(t_n)_n$ be an arbitrary sequence of real numbers. We have:

$e^{zt_n}(1 - F(t_n)) \geq \dfrac{1 - F(t_n)}{1 - F(t_n - 1)} \cdot (1 - F([t_n])) \cdot e^{z \cdot [t_n]} \xrightarrow[n \to \infty]{} \infty,$ hence the conclusion holds true.

In particular, we have: $M_x(z) = \displaystyle\int_0^\infty e^{zx} dF(x) = 1 + \int_0^\infty \int_0^x z \cdot e^{zy} dy dF(x) =$

$$= 1 + z \cdot \int_0^\infty \int_y^\infty e^{zy} dF(x) dy = 1 + z \cdot \int_0^\infty (1 - F(y)) e^{z \cdot y} dy = \infty,$$

as $e^{zy}(1 - F(y))$ is positive and not bounded.

In such cases, denoting by $B(x) = m_1^{-1} \cdot \displaystyle\int_0^x (1 - F(y)) dy$, the following result is obtained:

Proposition 3. If $B(x)$ is sub-exponential, then $\lim\limits_{r \to \infty} \dfrac{\Psi(r, \theta)}{1 - B(r)} = \lambda \cdot m_1 \cdot (g(\theta) - \lambda \cdot m_1)^{-1}.$

Risk reserve fund

Any insurance company must establish a risk reserve fund (R) so that the difference between the total amount of the claims (the total paid compensations) and the collected premiums exceeds this risk reserve with a probability less than an accepted value, α (the probability of ruin). We consider a model in which we have n policies of the same type, the paid damages for each policy being represented by random variable X with expected value m and variance σ^2. So, the amount of the total paid damages is $Y = \sum\limits_{i=1}^n X_i$,

where X_i are independent and identically distributed (with X) random variables. Also, we consider that the tariff system is established on the mean value principle, so the total net premium is $Pnt = M(Y) = n \cdot m$. The reserve fund R is defined through the relationship $P(Y - Pnt > R) \leq \alpha$. Using the Central Limit Theorem, we get $R \geq \sigma \cdot \sqrt{n} \cdot z_{1-\alpha}$, where $z_{1-\alpha}$ is the quantile of order $1 - \alpha$ of the $N(0,1)$ normal distribution. We take the minimum reserve of risk (denoted $R_{\min}^{TLC}$) $R_{\min}^{TLC} = \sigma \cdot \sqrt{n} \cdot z_{1-\alpha}$. In particular, if $X : \begin{pmatrix} S & 0 \\ p & 1-p \end{pmatrix}$, where p is probability of the occurrence of a damage and S is the insured sum, we get $R_{\min}^{TLC} = S \cdot \sqrt{n \cdot p \cdot (1-p)} \cdot z_{1-\alpha}$. Also, we can find the risk reserve using Chebyshev's inequality $P(|Y - M(Y)| \geq R) \leq \dfrac{D(Y)}{R^2}$. We have $P(Y - M(Y) > R) \leq P(|Y - M(Y)| \geq R)$. With $\dfrac{D(Y)}{R^2} \leq \alpha$,

then $R \geq \sigma \cdot \sqrt{\dfrac{n}{\alpha}}$. In this particular case, we obtain $R_{\min}^{Cheb} = S \cdot \sqrt{\dfrac{n \cdot p \cdot (1-p)}{\alpha}}$. Some numerical results are given in the next tables.

Table 1 **Risk reserve for** $p = 0,14$

α	0,005	0,01	0,05	0,1
$R_{\min}^{TLC}$	895,226.54	807,438.82	570,793.67	444,837.37
$R_{\min}^{Cheb}$	4,907,137.66	3,469,870.31	1,551,773.18	1,097,269.34

Table 2 **Risk reserve for** $p = 0,08$

α	0,005	0,01	0,05	0,1
$R_{\min}^{TLC}$	699,936.45	631,299.27	446,277.31	347,797.88
$R_{\min}^{Cheb}$	3,836,665.22	2,712,931.99	1,213,260.07	857,904.42

We notice that *TLC* method gives better results.

3. Numerical results and conclusions

For an exponential individual claim, then $\dfrac{\partial \Psi(r,\theta,\alpha,\lambda)}{\partial \theta} = -\dfrac{\lambda}{\alpha} \cdot \dfrac{1}{1+\theta} \cdot h(\theta) \cdot (1 + \lambda \cdot r \cdot h(\theta)) \cdot e^{-(\alpha - \lambda \cdot h(\theta)) \cdot r}$.

Hence, $\Psi(r,\theta,\alpha,\lambda)$ is decreasing with respect to θ . For $\theta = 0$, it follows $\Psi(r,0,\alpha,\lambda) = 1$ (therefore, if the premiums are not loaded, the initial reserve being disregarded, the ruin will appear certainly). For $\theta \to \infty$, it follows $\lim\limits_{\theta \to \infty} \Psi(r,\theta,\alpha,\lambda) = 0$. Obviously, this is only a good mathematical result, because the premiums can not be loaded as much as possible!

As $\Psi(r,\theta,\alpha,\lambda) = \dfrac{1}{1+\theta} \cdot e^{-\alpha \cdot \frac{\theta}{1+\theta} \cdot r}$, we notice that the ruin probability is constant with respect to the intensity of the claims number process. This fact does not seem incredible, but it is explicable if we analyze the model hypotheses, where the inputs (the cashed premiums) are found according to the mean value principle, being in this way related to the mean output flows (the compensation for damages). Therefore, the model provides the proportionality between the input and output cash. This fact can be considered restrictive against the real situation, where, the explosive growth of compensations for claims is not attended by a corresponding growth of the cashed premiums amount.

The ruin probability is decreasing also with respect to the parameter of the individual claim distribution.

We have: $\lim\limits_{\alpha \to 0} \Psi(r,\theta,\alpha,\lambda) = \dfrac{1}{1+\theta}$ and $\lim\limits_{\alpha \to \infty} \Psi(r,\theta,\alpha,\lambda) = \dfrac{1}{1+\theta} \cdot e^{-\infty} = 0.$

The first bound shows the behavior of the ruin probability when the expected value of the individual claim $\left(EX = \dfrac{1}{\alpha} \right)$ tends to grow as much as possible. If the load factor of premiums is strictly positive, then the ruin is not sure, but it has a high probability, inversely proportional to the value of the loading factor. In the case when the mean individual claim becomes negligible, the probability that the ruin will occur is practically null, in the conditions of the existence of an initial reserve and of the premium loading factor, fact pointed out by the second bound.

If we take the initial capital proportional to the value of the mean individual claim, (k – the proportionality factor), we obtain in the next table values for the ruin probability:

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Table 3 Ruin probability for some θ and k

θ/k	$1+1/\theta$	2	5	10	20	40
0.10	0.33444	0.75796	0.57703	0.36626	0.14756	0.02395
0.15	0.31990	0.66990	0.45297	0.23596	06403	0.00471
0.20	0.30657	0.59711	0.36217	0.15740	0.02973	0.00106
0.30	0.28298	0.48486	0.24263	0.07653	0.00761	0.00008
0.60	0.22992	0.29523	0.09585	0.01470	0.00035	$0.19\cdot10^{-6}$
0.80	0.20438	0.22840	0.06020	0.00652	0.00008	10^{-8}
1.00	0.18394	0.18394	0.04104	0.00337	0.00002	10^{-9}

Placing the adjustment coefficient R between $R_2 = \dfrac{\alpha^2}{1+\alpha}\cdot \ln(1+\theta)$ and $R_1 = \alpha\cdot\theta$, the interval of values

for the ruin probability has the bounds Ψ_1 and Ψ_2, see Table 4.

Table 4 Ruin probability and its margins

α	θ	r	R_1	R_2	Ψ	Ψ_1	Ψ_2
0.01	0.1	5000	0.001	$0.94\cdot10^{-5}$	0.00965	0.00674	0.95391
0.05	0.2	1000	0.01	0.00043	0.00020	0.000045	0.64784
0.05	0.2	5000	0.01	0.00043	$6.7\cdot10^{-19}$	$1.9\cdot10^{-22}$	0.11412
0.05	0.3	1000	0.015	0.00062	$0.74\cdot10^{-5}$	$0.3\cdot10^{-6}$	0.5354
0.05	0.5	1000	0.025	0.00096	$0.38\cdot10^{-7}$	$1.4\cdot10^{-11}$	0.3808
0.1	0.2	1000	0.02	0.00165	$0.48\cdot10^{-7}$	$2\cdot10^{-8}$	0.19062
0.1	0.3	500	0.03	0.00238	$0.75\cdot10^{-5}$	$0.3\cdot10^{-6}$	0.3034
0.5	0.2	500	0.1	0.0303	$6.7\cdot10^{-19}$	$1.9\cdot10^{-22}$	$0.25\cdot10^{-6}$
1	0.2	200	0.2	0.09116	$2.78\cdot10^{-15}$	$4.2\cdot10^{-18}$	$0.12\cdot10^{-7}$

Table 5 Ruin probability for some a and b

a	b	R_1	Ψ_1
0.25	0.25	0.72	$0.55\cdot10^{-6}$
0.25	0.75	0.96	$0.4\cdot10^{-8}$
0.5	0.5	0.8	$0.112\cdot10^{-6}$
0.5	0.25	0.7	$0.83\cdot10^{-6}$
0.5	0.75	0.9	$0.15\cdot10^{-7}$

If $X \sim Beta(a,b)$, taking $R_1 = 2\theta\cdot\dfrac{a+b+1}{a+1}$, $R_2 = \ln(1+\theta)$, $\theta = 0.3$ and $r = 20$ m. u. (monetary

units), we obtain $\Psi_2 = 0.00526$ and for Ψ_1 we give the values in Table 5.

The estimation of the ruin probability proposed by De Vylder [3] consists in the approximation of the surplus process $\{C(t)\}_{t\geq0}$ by a process $\{\tilde{C}(t)\}_{t\geq0}$ given by $\tilde{C}(t) = r + \tilde{c}\,t - \tilde{D}(t), t \geq 0$, where the aggregate damage paid process $\{\tilde{D}(t)\}_{t\geq0}$ is a compound Poisson process with parameter $\tilde{\lambda}$, and the distribution of the individual claim is $X \in Exp(\tilde{\alpha})$. The parameters of the new process are chosen such that the first three moments of $C(t)$ and $\tilde{C}(t)$ must be equal.

$$\begin{cases} E(C(t)) = E(\tilde{C}(t)) \\ E(C^2(t)) = E(\tilde{C}^2(t)) \\ E(C^3(t)) = E(\tilde{C}^3(t)) \end{cases} \Rightarrow \tilde{\alpha} = \frac{3m_2}{m_3},\ \tilde{\lambda} = \frac{9\lambda(m_2)^3}{2(m_3)^2},\ \tilde{c} = c - \lambda m_1 + \frac{\tilde{\lambda}}{\tilde{\alpha}},\ m_k = E(X^k), k = 1,2,3.$$

The estimation of the ruin probability is $\psi(r) = \dfrac{\tilde{\lambda}}{\tilde{\alpha} \cdot \tilde{c}} \cdot e^{-r\left(\tilde{\alpha} - \frac{\tilde{\lambda}}{\tilde{c}}\right)}$.

Numerical example: For a portfolio of car insurance there have been recorded $n = 100$ values of claims for the damage payments $x_1, x_2, ..., x_n$. The results for the first three empirical moments and other empirical values are $\hat{m}_1 = \dfrac{1}{n}\sum_{i=1}^{n} x_i = 2146.02$, $\hat{m}_2 = \dfrac{1}{n}\sum_{i=1}^{n} (x_i)^2 = 14\,313\,879.42$,

$\hat{m}_3 = \dfrac{1}{n}\sum_{i=1}^{n} (x_i)^3 = 158\,056\,031\,294.70$, the variance $s^2 = \dfrac{1}{n-1}\sum_{i=1}^{n}(x_i - \hat{m}_1)^2 = \dfrac{n}{n-1}\cdot\left(\hat{m}_2 - (\hat{m}_1)^2\right) = 9\,806\,543,01$,

the standard deviation $s = \sqrt{s^2} = 3131,54$, the coefficient of variation $v = \dfrac{s}{\hat{m}_1} = 1,4592$ and the skewness

$g = \dfrac{\frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{m}_1)^3}{\sqrt{\left(\frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{m}_1)^2\right)^3}} = \dfrac{\hat{m}_3 - 3\cdot\hat{m}_1\cdot\hat{m}_2 + 2\cdot(\hat{m}_1)^3}{\left(\hat{m}_2 - (\hat{m}_1)^2\right)^{3/2}} = 2,8320$ .These values suggest that the data are generated by a distribution law having a positive skewness, so the normal or the uniform distribution is not suitable. As the empirical coefficient of variation differs from 1 and the empirical skewness differs from 2, one can expect that the data are not generated from an exponential distribution, but distributions like Pareto, Gamma or lognormal can not be excluded.

Next we try to find out if the data are drown from a Pareto distribution. The probability density function of the individual claim $X \in Par(\alpha, \beta), \alpha > 0, \beta > 0$ is $f_X(x) = \dfrac{\beta}{\alpha}\cdot\left(\dfrac{\alpha}{\alpha + x}\right)^{\beta+1}, x > 0$, and the cumulative distribution function is $F_X(x) = P(X \le x) = 1 - \left(\dfrac{\alpha}{\alpha + x}\right)^{\beta}, x \ge 0$. The first three initial moments of X exist, but with constraints on the value of the parameter β, so: $m_1 = E(X) = \dfrac{\alpha}{\beta - 1}$, if $\beta > 1$, $m_2 = E(X^2) = \dfrac{2!\cdot\alpha^2}{(\beta-1)\cdot(\beta-2)}$, if $\beta > 2$, and, if $\beta > 3$, $m_3 = E(X^3) = \dfrac{3!\cdot\alpha^3}{(\beta-1)\cdot(\beta-2)\cdot(\beta-3)}$.

We estimate the parameters α and β by the method of moments, solving the system of equations $\begin{cases} E(X) = \hat{m}_1 \\ E(X^2) = \hat{m}_2 \end{cases}$. We get $\hat{\beta} = \dfrac{2\cdot\left(\hat{m}_2 - (\hat{m}_1)^2\right)}{\hat{m}_2 - 2\cdot(\hat{m}_1)^2}$ and $\hat{\alpha} = (\hat{\beta} - 1)\cdot\hat{m}_1 = \dfrac{\hat{m}_1\cdot\hat{m}_2}{\hat{m}_2 - 2\cdot(\hat{m}_1)^2}$. For the data we analyze we observe that $\hat{m}_2 > 2\cdot(\hat{m}_1)^2$, so we obtain the estimated values $\hat{\beta} = 3.8050$ and $\hat{\alpha} = 6019.48$. Further, we test the null hypothesis that $X \in Par(6019.48; 3.8050)$ with the help of Kolmogoroff goodness-of-fit test and we accept it (or we do not reject it). So, if $X \in Par(6019.48; 3.8050)$, then, $m_1 = 2145.982175$, $m_2 = 14313237.43$, $m_3 = 3.21087 \cdot 10^{11}$.

For $\lambda = 100$, $\theta = 0.25$, $c = 268247.7718$, we compute $\tilde{\alpha} = 0.000133732$, $\tilde{\lambda} = 12.79916651$, $\tilde{c} = 149356.7926$. For different levels of the initial capital (expressed in monetary units) of the insurer, we list the estimated ruin probabilities $\psi(80000) = 0.013732043$, $\psi(100000) = 0.005253987$, $\psi(150000) = 0.000475744$.

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