THE FUZZY OPTIMISTIC-REASONABLE-PESSIMISTIC INVENTORY MODEL

Lidia VESA
Doctoral School of Economic Sciences, Faculty of Economic Sciences, University of Oradea, Romania
lidiavesa@gmail.com

Abstract: *In inventory and production decision problems, decision makers are interested to identify the optimal inventory and production level. In a certain decision environment, the optimal inventory level could be determined through traditional inventory methods and the optimal ptodution level could be determined through linear programming algorithms. In an uncertain decision environment, the traditional methods and algorithms can not provide efficient and relevant solutions for these levels, due to the vague and changing parameters. In this case it is neccesary to develop new methods and models that can deal with vague variables and provide optimal levels. In this paper, the optimal inventory and production levels are determined through a single model that uses fuzzy linear programming. This new model is Fuzzy Optimistic-Reasonable-Pessimistic Inventory Model. It has three scenario: optimistic, reasonable and pessimistic, that are defined through triangular fuzzy numbers. In this way, decision makers can deal with vague parameters. These scenarios help managers to divide the Fuzzy ORP Model into three sub-models, that can be easily solved through traditional Simplex Algorithms. Each sub-model provides a crisp solution for each scenario. The solutions forms the final fuzzy optimal solution. The Fuzzy PRO Inventory Model helps managers to identify three optimal levels and to rank them according to their evaluations. This is useful, also, in predictions, where the decision makers should predict different scenarios for the production process. The limit of this model is the definition of the variables and scenarios. This model consider that all values for all variables and coefficients have the same definition: the inferior limit is related to the optimistic sceanrio, the peak is represents the reasonable limit and the superior limit is related to the pessimistic scenario. In real problem, the decision variables could have different definition than coefficients. The inferior limit of the cost is related to the optimistic scenario, but the superior limit of the production level can be related to the optimistic scenario. There are different representations for the scenarios.*

Keywords: *fuzzy number, simplex method, decision process, scenarios, inventory, production, level*

JEL Classification: *C44; C53; D24; M11.*

1. Introduction

"All organizations are at least 50 per cent waste – waste people, waste effort, waste space and waste time". (Robert Townsend, 1970, as cited in Waters, 2009) This waste could be reduced by combining lean and agile strategies in decision making process. A lean strategy means a detailed analysis of tactical operations and helps

decision makers: to remove operations that add no value, to eliminate delays, to reduce complexity, to simplify movements, to increase efficiency and therefore to eliminate the waste of resources. An agile strategy helps decison makers firstly, to keep a close look on consumers changing needs and secondly to react quickly to changes by providing improved services and products. There could be some problems if the managers take only a lean or agile approach. In inventory management, the lean strategy assume the idea that reducing stock would allow managers to save money and to minimize the inventory cost. On the contrary, the agile strategy support the idea that the companies should respond quickly to the costumers needs, by holdin stocks. Therefore, the decision makers should ensure an optimum inventory level that would satisfy the costumers need any time. If there are stockouts, the reorder cost would be higher and the costumers satisfaction would be low due to delivery-delay. The problem for the decision makers is to identify the optimum level of inventory that minimize the total cost of production and can be easily adapted to the changes of the consumers needs.
There are different methods that can be used in order to manage inventories efficiently. Economic Order Quantity, ABC Analysis, Just in time, Reorder point, Simplex method are just a few. These methods provides relevant solutions for inventory problems with known parameters. If the decision problem has some vague information or unknown parameters, the traditional inventory methods are less efficient. In an uncertain environment, they can be improved with fuzzy logic.
This paper aims to develop a new fuzzy linear model that combine traditional inventory and production optimization method, Dual Simplex Method, with Fuzzy logic. The linear model has fuzzy variables and coefficients. It is designed to have three scenarios: pesimistic, reasonable and optimistic. The scenarios are described through triangular fuzzy numbers. Therefore, the scenarios are considered for every variable and for every coefficient from the objective function and restrictions. The fuzzy model is divided in three sub-models associated with the scenarios. Each sub-model is solved through Dual Simplex Method. The new model is called in this paper: ORP Fuzzy Model (Optimistic-Reasonable- Pessimistic Fuzzy Model). This model is applied in an inventory and production planning decision problem.

2. Short literature review

In inventory decision problems, decision makers have three options to use or create models in order to obtain the optimum solution for the inventory level or production level. They can use pre-existing models, if these models fulfil the requirements of the decision problem. They can adapt the pre-existing models to the requirements. They can create new models, if the decision problem has some characteristics that are not considered on the pre-existing models.
In this paper, the need for building the new fuzzy model with three scenario comes from: the lack of the pre-existing models that can provide different approach for the decision maker’s objectives, the lack of the algorithms that can solve quickly and easy fully fuzzy linear models, the lack of using Non-fuzzy Simplex Algorithm to

solve fuzzy decision problems and the complexity of applying Fuzzy Algorithms to real decision problems. There are no software applications that can solve quickly the fuzzy problems through Simplex Algorithm. Therefore, the new model is designed to provide optimal solution to the fuzzy inventory and production problems, through the simplest way possible.

In the literature, the authors were more focused to develop new algorithms and methods to solve some linear models. This is emphasized in the following table:

Table 1 The literature review of fuzzy linear models

Authors	Fuzzy linear model	Methodology
Kumar et al. (2010)	$\begin{cases} \min \tilde{z} \approx \sum_{j=1}^{n} (p_j, q_j, r_j) \otimes (x_j, y_j, z_j) \\ \sum_{j=1}^{n} (a_{ij}, b_{ij}, c_{ij}) \otimes (x_j, y_j, z_j) \\ \preccurlyeq (b_i, g_i, h_i), i = \overline{1,m} \\ (x_j, y_j, z_j) \geq (0,0,0) \end{cases}$	The authors developed the fully fuzzy linear model and solved the model through ranking functions and α- level sets.
Cheng et al (2013)	$\begin{cases} \max \tilde{z} \approx \sum_{j=1}^{n} \left(c_j^{(1)} x_j^{(1)}, c_j^{(2)} x_j^{(2)}, c_j^{(3)} x_j^{(3)}\right) \\ \sum_{j=1}^{n} \left(a_{ij}^{(1)} x_j^{(1)}, a_{ij}^{(2)} x_j^{(2)}, a_{ij}^{(3)} x_j^{(3)}\right) \\ \preccurlyeq \left(b_i^{(1)} + p_i^{(1)}, b_i^{(2)} + p_i^{(2)}, b_i^{(3)} + p_i^{(3)}\right) \\ \sum_{j=1}^{n} \left(a_{ij}^{(1)} x_j^{(1)}, a_{ij}^{(2)} x_j^{(2)}, a_{ij}^{(3)} x_j^{(3)}\right) \\ \preccurlyeq \left(b_i^{(1)} - q_i^{(3)}, b_i^{(2)} - q_i^{(2)}, b_i^{(3)} - q_i^{(1)}\right) \\ \left(x_j^{(1)}, x_j^{(2)}, x_j^{(3)}\right) \succcurlyeq (0,0,0) \end{cases}$	The authors proposed a more flexible linear model. They added a fuzzy number p to right-hand side term and substracted a fuzzy number q from right-hand side term . These fuzzy numbers are the minimum and maximum values that right-hand side term could have.
Khan et. al (2013)	$\begin{cases} \min \tilde{z} = \tilde{c}^t \tilde{x} \\ \tilde{A}\tilde{x} \approx \tilde{b} \\ \tilde{x} \succcurlyeq 0 \end{cases}$	The authors proposed a linear model with three scenario objective function: pessimistic, reasonable and optimistic scenarios. They used membership functions in order to create the pessimistic and optimistic objective function.
Borzabadi și Alemy (2015)	$\min \tilde{z} = \tilde{c}^t \tilde{x}$ $\tilde{A}\tilde{x} = \tilde{b}$ $\tilde{x} \geq 0$	The authors proposed Dual Simplex Method to solve a simple linear fuzzy model. They used triangular fuzzy numbers.

Nasseri and Mahmoudi (2019)	$\begin{cases} \max \tilde{z} \approx \sum_{j=1}^{n} \left(c_j{}^l, c_j{}^c, c_j{}^u\right) \otimes \left(x_j{}^l, x_j{}^c, x_j{}^u\right) \\ \sum_{j=1}^{n} \left(A_{ij}{}^l, A_{ij}{}^c, A_{ij}{}^u\right) \otimes \left(x_j{}^l, x_j{}^c, x_j{}^u\right) \preccurlyeq \left(b_i{}^l, b_i{}^c, b_i{}^u\right) \\ \left(x_j{}^l, x_j{}^c, x_j{}^u\right) \succcurlyeq (0,0,0) \end{cases}$	The authors proposed a new method to transform a fuzzy model in crisp model. They transformed the objective function through ranking functions.
Boloș et. al (2020)	$\begin{cases} \mathrm{Z} = \overline{Z_B} - \sum_{j \in J_S}^{n} \left(Z_j^B - C_{aj}\right) x_j \\ X_B = \overline{X_B} - \sum_{j=1}^{n} \overline{C_{ij}^B}\, x_j \\ x_i = \overline{x_j^B} - \sum_{j=1}^{n} \overline{C_{ij}^B}\, x_j \\ x_i \geq 0, i \in \overline{1,n} \end{cases}$	The authors proposed a model that emphasize the relations between fuzzy basis variables (X_B) and crisp values of the variables (x_i). The model was solved through Fuzzy Primal Simplex Algorithm.
Ghoushchi et al. (2021)	$\begin{cases} \max \tilde{z} \approx \sum_{j=1}^{n} \left(c_j{}^l, c_j{}^m, c_j{}^u\right) \otimes \left(x_j{}^l, x_j{}^m, x_j{}^u\right) \\ \sum_{j=1}^{n} \left(A_{ij}{}^l, A_{ij}{}^m, A_{ij}{}^u\right) \otimes \left(x_j{}^l, x_j{}^m, x_j{}^u\right) \preccurlyeq \left(b_i{}^l, b_i{}^m, b_i{}^u\right) \\ \left(x_j{}^l, x_j{}^m, x_j{}^u\right) \succcurlyeq (0,0,0) \end{cases}$	They proposed a new model with modified triangular fuzzy numbers. These modified fuzzy numbers were developed using alpha-cut theory. Through this model the uncertainty is consistently reduced.
Davoodi and Rahman (2021)	$\begin{cases} \min \Re\left(\sum_{j=1}^{n} \tilde{c}_j \tilde{x}_j\right) \\ \Re\left(\sum_{j=1}^{n} \tilde{a}_{ij} \tilde{x}_j\right) \geq \Re\left(\tilde{b}_i\right), i = 1,2,\dots,m \\ \frac{\alpha_{xj}}{\lvert m_{xj} \rvert} \leq M, j = 1,2,\dots,n \\ \frac{\beta_{xj}}{\lvert m_{xj} \rvert} \leq M, j = 1,2,\dots,n \\ \Re\left(\tilde{x}_j\right) \geq 0, j = 1,\dots,n \end{cases}$	The authors proposed a new linear model. In this model a paramter is used in order to determine the maximum and minimum values for fuzzy variables. This parameter helps managers to reduce the universe discours of the fuzzy solution. In real-life situations, the managers would be able to understand and implement more easily a fuzzy solution with a reduced numbers of crisp values.

All these models are fully fuzzy linear models that can deal with triangular fuzzy numbers. Kumar et al (2010) applied the linear model in production planning problem. Khan et al (2013) proposed an example of his model in project selection problems. Bolos et al (2020) uses his model to support decision makers in investment decisions in tangible assets. Davoodi and Rahman (2021) applied his model in a farm planning problem.

3. Pessimistic-Reasonable-Optimistic (ORP) Fuzzy Linear Model

3.1. The Fuzzy Operations of ORP Model

The ORP Model is a fuzzy linear model that uses triangular fuzzy numbers for all variables and coefficients and is developed through fuzzy arithmetic operations between triangular fuzzy numbers.

Let A and B be two triangular fuzzy number, represented by three points: $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ and by the following membership functions:

$$\mu_{(A)}(x) = \begin{cases} 0 \quad , \quad x < a_1 \quad , x > a_3 \\ \frac{x-a_1}{a_2-a_1} \quad , \quad a_1 < x < a_2 \\ \frac{a_3-x}{a_3-a_2} \quad , \quad a_2 < x < a_3 \\ 1 \quad , \quad x = a_2 \end{cases} \qquad \mu_{(B)}(x) = \begin{cases} 0 \quad , \quad x < b_1 \quad , x > b_3 \\ \frac{x-b_1}{b_2-b_1} \quad , \quad b_1 < x < b_2 \\ \frac{a_3-x}{a_3-a_2} \quad , \quad b_2 < x < b_3 \\ 1 \quad , \quad x = b_2 \end{cases}$$

Considering these triangular fuzzy numbers, the arithmetic operations defined by Borzabadi and Alemy (2015) on these numbers are:

- Addition
  $$\tilde{A} \oplus \tilde{B} = (a_1, a_2, a_3) \oplus (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$
- Subtraction;
  $$\tilde{A} \ominus \tilde{B} = (a_1, a_2, a_3) \ominus (b_1, b_2, b_3) = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$$
- Multiplication;
  $$\tilde{A} \otimes \tilde{B} = (a_1, a_2, a_3) \otimes (b_1, b_2, b_3) = (a_1 b_1, a_2 b_2, a_3 b_3)$$
- Division.
  $$\tilde{A} \oslash \tilde{B} = (a_1, a_2, a_3) \oslash (b_1, b_2, b_3) = (\frac{a_1}{b_3}, \frac{a_2}{b_2}, \frac{a_3}{b_1})$$

3.2. The Fuzzy Elements of ORP Model

The ORP Model is a linear programming model that has the following elements: variables, coefficients, restrictions and right-hand side terms. Due to the minimization objective function, the inferior values of the parameters would be considered optimistic values and the superior values of the parameters would be considered pessimistic values.

$$\min \tilde{z} \approx \sum_{j=1}^{n} \left(c_j{}^o, c_j{}^r, c_j{}^p\right) \otimes \left(x_j{}^o, x_j{}^r, x_j{}^p\right)$$
$$\sum_{j=1}^{n} \left(a_{ij}{}^o, a_{ij}{}^r, a_{ij}{}^p\right) \otimes \left(x_j{}^o, x_j{}^r, x_j{}^p\right) \succcurlyeq (b_i{}^o, b_i{}^r, b_i{}^p)$$
$$\left(x_j{}^p, x_j{}^r, x_j{}^o\right) \succcurlyeq (0,0,0)$$

Where: $\left(c_j{}^o, c_j{}^r, c_j{}^p\right)$ – triangular fuzzy coefficients of objective function with three scenarios:

optimistic, reasonable and pessimistic

$\left(x_j{}^o, x_j{}^r, x_j{}^p\right)$ – traingular fuzzy variables with three scenarios:

optimistic, reasonable and pessimistic

$\left(a_{ij}{}^o, a_{ij}{}^r, a_{ij}{}^p\right)$ – traingular fuzzy coefficients of constraints with three sceanrios:

optimistic, reasonable and pessimistic

$(b_i{}^o, b_i{}^r, b_i{}^p)$ – triangular fuzzy right-hand side terms with three secanrios:

optimistic, reasonable and pessimistic

Remark 1: Let *x* be a crisp value within triangular fuzzy coefficients. This value is a optimistic value if $c_j^o < x < c_j^r$, an pessimistic value if $c_j^r < x < c_j^p$ and a reasonable value if $x = c_j^r$.

Remark 2: Let *y* be a crisp value within triangular fuzzy variables. This value is a optimistic value if $x_j{}^o < y < x_j{}^r$, an pessimistic value if $x_j{}^r < y < x_j{}^p$ and a reasonable value if $y = x_j{}^r$.

Remark 3: Let *z* be a crisp value within triangular fuzzy coefficients of constraints. This value is a optimistic value if $a_{ij}{}^o < z < a_{ij}{}^r$, an pessimistic value if $a_{ij}{}^r < z < a_{ij}{}^p$ and a reasonable value if $z = a_{ij}{}^r$.

Remark 4: Let *w* be a crisp value within triangular fuzzy coefficients of constraints. This value is a optimistic value if $b_i{}^o < w < b_i{}^r$, an pessimistic value if $b_i{}^r < w < b_i{}^p$ and a reasonable value if $w = b_i{}^r$.

3.3. The Transformation of Fuzzy ORP Model into Crisp ORP Models

In order to solve the linear model, it is necessary transform this fuzzy model into a crisp model. Nasseri and Mahmoudi (2019) transformed this model through fuzzy ranking functions. This fuzzy ORP model is transformed into crisp ORP model through fuzzy arithmetic operations:

$$\min \tilde{z} \approx \sum_{j=1}^{n} \left(c_j{}^o, c_j{}^r, c_j{}^p\right) \otimes \left(x_j{}^o, x_j{}^r, x_j{}^p\right)$$

$$\sum_{j=1}^{n} \left(a_{ij}{}^o, a_{ij}{}^r, a_{ij}{}^p\right) \otimes \left(x_j{}^o, x_j{}^r, x_j{}^p\right) \succcurlyeq (b_i{}^o, b_i{}^r, b_i{}^p)$$

$$\left(x_j{}^o, x_j{}^r, x_j{}^p\right) \succcurlyeq (0,0,0)$$

$$\min \tilde{z} \approx \sum_{j=1}^{n} \left(c_j{}^o x_j{}^o, c_j{}^r x_j{}^r, c_j{}^p x_j{}^p\right)$$

$$\Longrightarrow \qquad \sum_{j=1}^{n} \left(a_{ij}{}^o x_j{}^o, a_{ij}{}^r x_j{}^r, a_{ij}{}^p x_j{}^p\right) \succcurlyeq (b_i{}^o, b_i{}^r, b_i{}^p)$$

$$\left(x_j{}^p, x_j{}^r, x_j{}^o\right) \succcurlyeq (0,0,0)$$

This modified model is divided in three sub-models, as follows:

- Optimistic sub-model:

$$\min z = \sum_{j=1}^{n} c_j{}^o x_j{}^o$$

$$\sum_{j=1}^{n} a_{ij}{}^o x_j{}^o \geq \; b_i{}^o$$

$$x_j{}^o \geq 0$$

- Reasonable sub-model:

$$\min z = \sum_{j=1}^{n} c_j{}^r x_j{}^r$$

$$\sum_{j=1}^{n} a_{ij}{}^r x_j{}^r \geq \; b_i{}^r$$

$$x_j{}^r \geq 0$$

- Pessimistic sub-model:

$$\min z \approx \sum_{j=1}^{n} c_j{}^p x_j{}^p$$

$$\sum_{j=1}^{n} a_{ij}{}^p x_j{}^p \geq \; b_i{}^p$$

$$x_j{}^p \geq 0$$

The solutions of these sub-models are considered the inferior limit, the peak and the superior limit of the triangular fuzzy solution $\left(x_j{}^o, x_j{}^r, x_j{}^p\right)$. It is possible only if the $x_j{}^o < x_j{}^r < x_j{}^p$. Therefore, this is only a rule in order to solve the Fuzzy PRO Model.

4. The Inventory Fuzzy PRO Model
4.1. The Elements of the Inventory Fuzzy PRO Model

In an inventory decision problem, the Fuzzy PRO Model was created considering the three scenarios presented in the previous section. In addition, the Fuzzy Inventory PRO Model has three types of variables: production quantity, inventory quantity and unfinished quantity. The Fuzzy Inventory Model has the following structure:
Fuzzy objective function:

$$\min \tilde{z} = \sum_{i=1}^{n} (\tilde{c}_i \otimes \tilde{q}_i + \tilde{h}_i \otimes \tilde{s}_i + \tilde{k}_i \otimes \tilde{u}_i)$$

Fuzzy restrictions:

$$\tilde{d}_i = \tilde{s}_{i-1} + \tilde{u}_{i-1} + \tilde{q}_i - \tilde{s}_i + \tilde{u}_i$$

$$\sum_{i=1}^{n} \tilde{a}_{ik} \otimes \tilde{q}_i \succcurlyeq \tilde{b}_k$$

Non-negativity constraints:

$$\tilde{q}_i, \tilde{s}_i, \tilde{u}_i \succ 0$$

Where:
$\tilde{c}_i = ({c_i}^o, {c_i}^r, {c_i}^p)$ – the production cost for product *i*
$\tilde{q}_i = ({q_i}^o, {q_i}^r, {q_i}^p)$ – the production quantity for product *i*
$\tilde{h}_i = ({h_i}^o, {h_i}^r, {h_i}^p)$ – the holding cost for product *i*
$\tilde{s}_i = ({s_i}^o, {s_i}^r, {s_i}^p)$ – the inventory quantity for product *i*
$\tilde{k}_i = ({k_i}^o, {k_i}^r, {k_i}^p)$ – the penalty cost for unfinished product *i*
$\tilde{u}_i = ({u_i}^o, {u_i}^r, {u_i}^p)$ – the unfinished quantity for product *i*
$\tilde{d}_i = ({d_i}^p, {d_i}^r, {d_i}^o)$ – the demand for product *i*
$\tilde{a}_{ik} = ({a_{ik}}^o, {a_{ik}}^r, {a_{ik}}^p)$ – the amount of k resource used for production of the product *i*
$\tilde{b}_k = ({b_k}^o, {b_k}^r, {b_k}^p)$ – available quantity for resource *k*

The fuzzy variables were defined trough the three scenarios: pessimistic, reasonable and optimistic. The variables were defined considering the minimization objective function, which is a minimization cost function. If the objective of this model is to minimize the production inventory and penalty costs, then the minimum value of these fuzzy costs would be the optimistic values and the maximum value of these fuzzy costs would be the pessimistic values of the fuzzy costs. Also, the minimum values of fuzzy variables, would be considered optimistic and the maximum values would be considered pessimistic values. The demand is defined different from the

costs and quantities, because the maximum demands are related to optimistic scenarios and minimum demands are related to pessimistic scenarios.
There could be another way to define the variables. They could be defined with minimum values as pessimistic values and with maximum values as optimistic values. In this case, multiplying the costs with quantities means to produce with minimum cost maximum quantities. This case is part of a larger research and it will be published in a subsequent papers.

4.2. The Elements of the Inventory Fuzzy PRO Model
The transformation of this model into a crisp one is realised using the arithmetic operations, as follows:

$$\min \tilde{z} = \sum_{i=1}^{n} [(c_i{}^o, c_i{}^r, c_i{}^p) \otimes (q_i{}^o, q_i{}^r, q_i{}^p) + (h_i{}^o, h_i{}^r, h_i{}^p) \otimes (s_i{}^o, s_i{}^r, s_i{}^p) + (k_i{}^o, k_i{}^r, k_i{}^p) \otimes (u_i{}^o, u_i{}^r, u_i{}^p)]$$

$$(d_i{}^p, d_i{}^r, d_i{}^o) = \left(s_{(i-1)}{}^o, s_{(i-1)}{}^r, s_{(i-1)}{}^p\right) - \left(u_{(i-1)}{}^o, u_{(i-1)}{}^r, u_{3(i-1)}{}^p\right) + (q_i{}^o, q_i{}^r, q_i{}^p) - (s_i{}^o, s_i{}^r, s_i{}^p) + (u_i{}^o, u_i{}^r, u_i{}^p)$$

$$\sum_{i=1}^{n} (a_{ik}{}^o, a_{ik}{}^r, a_{ik}{}^p) \otimes (q_i{}^o, q_i{}^r, q_i{}^p) \preccurlyeq (b_k{}^o, b_k{}^r, b_k{}^p)$$

$$(q_i{}^o, q_i{}^r, q_i{}^p), (s_i{}^o, s_i{}^r, s_i{}^p), (u_i{}^o, u_i{}^r, u_i{}^p) \succcurlyeq (0,0,0)$$

$$\Rightarrow \min \tilde{z} = \sum_{i=1}^{n} [(c_i{}^o \times q_i{}^o, c_i{}^r \times q_i{}^r, c_i{}^p \times q_i{}^p) + (h_i{}^o \times s_i{}^o, h_i{}^r \times s_i{}^r, h_i{}^p \times s_i{}^p) + (k_i{}^o \times u_i{}^o, k_i{}^r \times u_i{}^r, k_i{}^p \times u_i{}^p)]$$

$$(d_i{}^p, d_i{}^r, d_i{}^o) = \left(s_{(i-1)}{}^o - u_{3(i-1)}{}^p, s_{(i-1)}{}^r - u_{(i-1)}{}^r, s_{(i-1)}{}^p - u_{(i-1)}{}^o\right) + (q_i{}^o - s_i{}^p, q_i{}^r - s_i{}^r, q_i{}^p - s_i{}^o) + (u_i{}^o, u_i{}^r, u_i{}^p)$$

$$\sum_{i=1}^{n} (a_{ik}{}^o q_i{}^o, a_{ik}{}^r q_i{}^r, a_{ik}{}^p q_i{}^p) \preccurlyeq (b_k{}^o, b_k{}^r, b_k{}^p)$$

$$(q_i{}^o, q_i{}^r, q_i{}^p), (s_i{}^o, s_i{}^r, s_i{}^p), (u_i{}^o, u_i{}^r, u_i{}^p) \succcurlyeq (0,0,0)$$

$$\Rightarrow \min \tilde{z} = \sum_{i=1}^{n} [(c_i{}^o \times q_i{}^o + h_i{}^o \times s_i{}^o + k_i{}^o \times u_i{}^o, c_i{}^r \times q_i{}^r + h_i{}^r \times s_i{}^r + k_i{}^r \times u_i{}^r, c_i{}^p \times q_i{}^p + h_i{}^p \times s_i{}^p + k_i{}^p \times u_i{}^p)]$$

$$(d_i{}^p, d_i{}^r, d_i{}^o) = \left(s_{(i-1)}{}^o - u_{3(i-1)}{}^p + q_i{}^o - s_i{}^p + u_i{}^o, s_{(i-1)}{}^r - u_{(i-1)}{}^r + q_i{}^r - s_i{}^r + u_i{}^r, s_{(i-1)}{}^p - u_{(i-1)}{}^o + q_i{}^p - s_i{}^o + u_i{}^p\right)$$

$$\sum_{i=1}^{n}(a_{ik}{}^{o}q_i{}^{o}, a_{ik}{}^{r}q_i{}^{r}, a_{ik}{}^{p}q_i{}^{p}) \preccurlyeq (b_k{}^{o}, b_k{}^{r}, b_k{}^{p})$$

$$(q_i{}^{o}, q_i{}^{r}, q_i{}^{p}), (s_i{}^{o}, s_i{}^{r}, s_i{}^{p}), (u_i{}^{o}, u_i{}^{r}, u_i{}^{p}) \succcurlyeq$$

(0,0,0)

This transformed model is divided in three sub-models:

- Optimistic sub-model

$$\min z^{o} = \sum_{i=1}^{n}(c_i{}^{o} \times q_i{}^{o} + h_i{}^{o} \times s_i{}^{o} + k_i{}^{o} \times u_i{}^{o})$$
$$d_i{}^{p} = s_{(i-1)}{}^{o} - u_{3(i-1)}{}^{p} + q_i{}^{o} - s_i{}^{p} + u_i{}^{o}$$
$$\sum_{i=1}^{n} a_{ik}{}^{o}q_i{}^{o} \leq b_k{}^{o}$$
$$q_i{}^{o} \geq 0$$

- Reasonable sub-model

$$\min z^{r} = \sum_{i=1}^{n}(c_i{}^{r} \times q_i{}^{r} + h_i{}^{r} \times s_i{}^{r} + k_i{}^{r} \times u_i{}^{r})$$
$$d_i{}^{r} = s_{(i-1)}{}^{r} - u_{(i-1)}{}^{r} + q_i{}^{r} - s_i{}^{r} + u_i{}^{r}$$
$$\sum_{i=1}^{n} a_{ik}{}^{r}q_i{}^{r} \leq b_k{}^{o}$$
$$q_i{}^{r} \geq 0$$

- Pessimistic sub-model

$$\min z^{p} = \sum_{i=1}^{n}(c_i{}^{p} \times q_i{}^{p} + h_i{}^{p} \times s_i{}^{p} + k_i{}^{p} \times u_i{}^{p})$$
$$d_i{}^{o} = s_{(i-1)}{}^{p} - u_{(i-1)}{}^{o} + q_i{}^{p} - s_i{}^{o} + u_i{}^{p}$$
$$\sum_{i=1}^{n} a_{ik}{}^{p}q_i{}^{p} \leq b_k{}^{p}$$
$$q_i{}^{o} \geq 0$$

The solutions of the sub-models form the final fuzzy solution of the Fuzzy Inventory ORP Model. The sub-models are solved using Dual Fuzzy Algorithm.

4.3. Testing the Fuzzy Inventory PRO Model

A furniture company produces four types of products. The decision makers plan the production process every month and they need to have a linear model that can provide an optimum solution even in vague, uncertain and changing environment. The model they need is an Inventory ORP Model with 12 variables and with following structure:

Fuzzy Objective Function:

$\min \tilde{z} = (1798\ 2349\ 2773) \otimes ({q_1}^o\ {q_1}^r\ {q_1}^p) \oplus (2327\ 3195\ 4131) \otimes ({q_2}^o\ {q_2}^r\ {q_2}^p) \oplus (202\ 378\ 510) \otimes ({q_3}^o\ {q_3}^r\ {q_3}^p) \oplus (1020\ 1546\ 2372) \otimes ({q_4}^o\ {q_4}^r\ {q_4}^p) \oplus (2413\ 2477\ 2541) \otimes ({s_1}^o\ {s_1}^r\ {s_1}^p) \oplus (3437\ 3697\ 3921) \otimes ({s_2}^o\ {s_2}^r\ {s_2}^p) \oplus (459\ 540\ 691) \otimes ({s_3}^o\ {s_3}^r\ {s_3}^p) \oplus (1709\ 1872\ 2035) \otimes ({s_4}^o\ {s_4}^r\ {s_4}^p) \oplus (2349\ 2753\ 3057) \otimes ({u_1}^o\ {u_1}^r\ {u_1}^p) \oplus (3195\ 3713\ 4131) \otimes ({u_2}^o\ {u_2}^r\ {u_2}^p) \oplus (378\ 494\ 610) \otimes ({u_3}^o\ {u_3}^r\ {u_3}^p) \oplus (1546\ 2009\ 2472) \otimes ({u_4}^o\ {u_4}^r\ {u_4}^p)$

Fuzzy Restrictions:

$({q_1}^o\ {q_1}^r\ {q_1}^p) \ominus ({s_1}^o\ {s_1}^r\ {s_1}^p) \oplus ({u_1}^o\ {u_1}^r\ {u_1}^p) \succcurlyeq (20\ 47\ 68)$
$({q_2}^o\ {q_2}^r\ {q_2}^p) \ominus ({s_2}^o\ {s_2}^r\ {s_2}^p) \oplus ({u_2}^o\ {u_2}^r\ {u_2}^p) \succcurlyeq (19\ 28\ 45)$
$({q_3}^o\ {q_3}^r\ {q_3}^p) \ominus ({s_3}^o\ {s_3}^r\ {s_3}^p) \oplus ({u_3}^o\ {u_3}^r\ {u_3}^p) \succcurlyeq (40\ 60\ 80)$
$({q_4}^o\ {q_4}^r\ {q_4}^p) \ominus ({s_4}^o\ {s_4}^r\ {s_4}^p) \oplus ({u_4}^o\ {u_4}^r\ {u_4}^p) \succcurlyeq (15\ 35\ 60)$
$(38\ 41\ 45) \otimes ({q_1}^o\ {q_1}^r\ {q_1}^p) \oplus (75\ 80\ 86) \otimes ({q_2}^o\ {q_2}^r\ {q_2}^p) \oplus (25\ 28\ 31) \otimes ({q_3}^o\ {q_3}^r\ {q_3}^p) \oplus (37\ 40\ 45) \otimes ({q_4}^o\ {q_4}^r\ {q_4}^p) \preccurlyeq (5400\ 11000\ 17000)$
$(0.3\ 0.35\ 0.4) \otimes ({q_1}^o\ {q_1}^r\ {q_1}^p) \oplus (0.6\ 0.64\ 0.7) \otimes ({q_2}^o\ {q_2}^r\ {q_2}^p) \oplus (0.08\ 0.09\ 0.1) \otimes ({q_3}^o\ {q_3}^r\ {q_3}^p) \oplus (0.2\ 0.25\ 0.3) \otimes ({q_4}^o\ {q_4}^r\ {q_4}^p) \preccurlyeq (100\ 150\ 300)$
$(278\ 296\ 320) \otimes ({q_1}^o\ {q_1}^r\ {q_1}^p) \oplus (145\ 161\ 191) \otimes ({q_2}^o\ {q_2}^r\ {q_2}^p) \oplus (55\ 65\ 78) \otimes ({q_3}^o\ {q_3}^r\ {q_3}^p) \oplus (72\ 90\ 107) \otimes ({q_4}^o\ {q_4}^r\ {q_4}^p) \preccurlyeq (23000\ 34000\ 47000)$
$(1.9\ 2\ 2.2) \otimes ({s_1}^o\ {s_1}^r\ {s_1}^p) \oplus (1.7\ 1.85\ 1.95) \otimes ({s_2}^o\ {s_2}^r\ {s_2}^p) \oplus (1.3\ 1.45\ 1.60) \otimes ({s_3}^o\ {s_3}^r\ {s_3}^p) \oplus (1.4\ 1.6\ 1.8)) \otimes ({s_4}^o\ {s_4}^r\ {s_4}^p) \preccurlyeq (200\ 300\ 400)$
$({s_1}^o\ {s_1}^r\ {s_1}^p) \oplus ({s_2}^o\ {s_2}^r\ {s_2}^p) \oplus ({s_3}^o\ {s_3}^r\ {s_3}^p) \oplus ({s_4}^o\ {s_4}^r\ {s_4}^p) \succcurlyeq (60\ 80\ 100)$
$({u_1}^o\ {u_1}^r\ {u_1}^p) \oplus ({u_2}^o\ {u_2}^r\ {u_2}^p) \oplus ({u_3}^o\ {u_3}^r\ {u_3}^p) \oplus ({u_4}^o\ {u_4}^r\ {u_4}^p) \succcurlyeq (10\ 20\ 30)$

Where: - $({q_1}^o\ {q_1}^r\ {q_1}^p)$, $({s_1}^o\ {s_1}^r\ {s_1}^p), ({u_1}^o\ {u_1}^r\ {u_1}^p)$ – production, inventory and unfinished quantities for the first product

- $({q_2}^o\ {q_2}^r\ {q_2}^p)$, $({s_2}^o\ {s_2}^r\ {s_2}^p)$, $({u_2}^o\ {u_2}^r\ {u_2}^p)$ – production, inventory and unfinished quantities for the second product
- $({q_3}^o\ {q_3}^r\ {q_3}^p), ({s_3}^o\ {s_3}^r\ {s_3}^p), ({u_3}^o\ {u_3}^r\ {u_3}^p)$ – production, inventory and unfinished quantities for the third product
- $({q_4}^o\ {q_4}^r\ {q_4}^p)$, $({s_4}^o\ {s_4}^r\ {s_4}^p)$, $({u_4}^o\ {u_4}^r\ {u_4}^p)$ – production, inventory and unfinished quantities for the fourth product

This model is divided in three sub models in order to obtain the solution.

- Optimistic sub-model

$\min \tilde{z} = 1798 \times {q_1}^o + 2327 \times {q_2}^o + 202 \times {q_3}^o + 1020 \times {q_4}^o + 2413 \times {s_1}^o + 3437 \times {s_2}^o + 459 \times {s_3}^o + 1709 \times {s_4}^o + 2349 \times {u_1}^o + 3195 \times {u_2}^o + 378 \times {u_3}^o + 1546 \times {u_4}^o$
${q_1}^o - {s_1}^o + {u_1}^o \geq 20$
${q_2}^o - {s_2}^o + {u_2}^o \geq 19$
${q_3}^o - {s_3}^o + {u_3}^o \geq 40$
${q_1}^o - {s_1}^o + {u_1}^o \geq 15$
$38 \times {q_1}^o + 75 \times {q_2}^o + 25 \times {q_3}^o + 37 \times {q_4}^o \leq 5400$
$0.3 \times {q_1}^o + 0.6 \times {q_2}^o + 0.08 \times {q_3}^o + 0.2 \times {q_4}^o \leq 100$
$278 \times {q_1}^o + 145 \times {q_2}^o + 55 \times {q_3}^o + 72 \times {q_4}^o \leq 23000$
$1.9 \times {s_1}^o + 1.7 \times {s_2}^o + 1.3 \times {s_3}^o + 1.4 \times {s_4}^o \leq 200$

$s_1{}^o + s_2{}^o + s_3{}^o + s_4{}^o \geq 60$
$u_1{}^o + u_2{}^o + u_3{}^o + u_4{}^o \geq 10$

- Reasonable sub-model

$\min \tilde{z} = 2349 \times q_1{}^r + 3195 \times q_2{}^r + 378 \times q_3{}^r + 1546 \times q_4{}^r + 2477 \times s_1{}^r + 3697 \times s_2{}^r + 540 \times s_3{}^r + 1872 \times s_4{}^r + 2753 \times u_1{}^r + 3713 \times u_2{}^r + 494 \times u_3{}^r + 2009 \times u_4{}^r$
$q_1{}^r - s_1{}^r + u_1{}^r \geq 47$
$q_2{}^r - s_2{}^r + u_2{}^r \geq 28$
$q_3{}^r - s_3{}^r + u_3{}^r \geq 60$
$q_1{}^r - s_1{}^r + u_1{}^r \geq 35$
$41 \times q_1{}^r + 80 \times q_2{}^r + 28 \times q_3{}^r + 40 \times q_4{}^r \leq 11000$
$0.35 \times q_1{}^r + 0.64 \times q_2{}^r + 0.09 \times q_3{}^r + 0.25 \times q_4{}^r \leq 150$
$296 \times q_1{}^r + 161 \times q_2{}^r + 65 \times q_3{}^r + 90 \times q_4{}^r \leq 34000$
$2 \times s_1{}^r + 1.85 \times s_2{}^r + 1.45 \times s_3{}^r + 1.6 \times s_4{}^r \leq 300$
$s_1{}^r + s_2{}^r + s_3{}^r + s_4{}^r \geq 80$
$u_1{}^r + u_2{}^r + u_3{}^r + u_4{}^r \geq 20$

- Pessimistic sub-model

$\min \tilde{z} = 2773 \times q_1{}^p + 4131 \times q_2{}^p + 510 \times q_3{}^p + 2372 \times q_4{}^p + 2541 \times s_1{}^p + 3921 \times s_2{}^p + 691 \times s_3{}^p + 2035 \times s_4{}^p + 3057 \times u_1{}^p + 4131 \times u_2{}^p + 610 \times u_3{}^p + 2472 \times u_4{}^p$
$q_1{}^p - s_1{}^p + u_1{}^p \geq 68$
$q_2{}^p - s_2{}^p + u_2{}^p \geq 45$
$q_3{}^p - s_3{}^p + u_3{}^p \geq 80$
$q_1{}^p - s_1{}^p + u_1{}^p \geq 60$
$45 \times q_1{}^p + 86 \times q_2{}^p + 31 \times q_3{}^p + 45 \times q_4{}^p \leq 17000$
$0.4 \times q_1{}^p + 0.7 \times q_2{}^p + 0.1 \times q_3{}^p + 0.3 \times q_4{}^p \leq 300$
$320 \times q_1{}^p + 191 \times q_2{}^p + 78 \times q_3{}^p + 107 \times q_4{}^p \leq 47000$
$2.2 \times s_1{}^p + 1.95 \times s_2{}^p + 1.60 \times s_3{}^p + 1.8 \times s_4{}^p \leq 400$
$s_1{}^p + s_2{}^p + s_3{}^p + s_4{}^p \geq 100$
$u_1{}^p + u_2{}^p + u_3{}^p + u_4{}^p \geq 30$

The first sub-model has the following optimal solution:

$q_1{}^o = 20,\ q_2{}^o = 19,\ q_3{}^o = 90,\ q_4{}^o = 15, s_1{}^o = 0, s_2{}^o = 0,\ s_3{}^o = 60,\ s_4{}^o = 0, u_1{}^o = u_2{}^o = u_3{}^o = u_4{}^o = 0$.

The value of objective function in optimal solution is: $\min z = 144973$

The second sub-model has the following optimal solution:

$q_1{}^r = 47,\ q_2{}^r = 28,\ q_3{}^r = 120,\ q_4{}^r = 35, s_1{}^r = 0, s_2{}^r = 0,\ s_3{}^r = 60,\ s_4{}^r = 0, u_1{}^o = u_2{}^o = u_3{}^o = u_4{}^o = 0$

The value of objective function in optimal solution is: $\min z = 326873$

The third sub-model has the following optimal solution:

$q_1{}^p = 68,\ q_2{}^p = 38,\ q_3{}^p = 180,\ q_4{}^p = 49, s_1{}^p = 0, s_2{}^p = 0,\ s_3{}^p = 100,\ s_4{}^p = 0, u_1{}^p = 0,\ u_2{}^p = 7,\ u_3{}^p = 0,\ u_4{}^p = 22.$

The value of objective function in optimal solution is: $\min \mathrm{z} = 667679$

Combining the values of the three sub-models it results the following fuzzy solution:

$\tilde{q}_1 = (20\ 47\ 68), \tilde{s}_1 = (0\ 0\ 0), \tilde{u}_1 = (0\ 0\ 0)$ – for the first product;

$\tilde{q}_2 = (19\ 28\ 38)$, $\tilde{s}_2 = (0\ 0\ 0)$, $\tilde{u}_2 = (0\ 0\ 7)$ – for the second product;
$\tilde{q}_3 = (90\ 120\ 180)$, $\tilde{s}_3 = (60\ 60\ 100)$, $\tilde{u}_3 = (0{,}0{,}0)$ – for the third product;
$\tilde{q}_4 = (15\ 35\ 49)$, $\tilde{s}_4 = (0{,}0{,}0)$, $\tilde{u}_1 = (0\ 0\ 22)$ – for the fourth product.

The fuzzy value of objective function is: $\min \tilde{z} = (144973, 326873, 667679)$. The solution is optimal becuause the fuzzy numbers that represents the solutions respect the rule : ${x_j}^o < {x_j}^p < {x_j}^r, j = \overline{1,4}$, for each product.

Conclusions

In modelling and testing Fuzzy Inventory ORP Model, the following strengths and weaknesses can be identified:

Strengths:

- The Fuzzy Inventory ORP Model allows decision-makers to formulate decision problems, even if the parameters are vague;
- The Fuzzy Inventory ORP Model allows managers to identify the optimal production and inventory level even in uncertain conditions;
- The Fuzzy Inventory ORP Model provides a flexible optimal solution, which can be easily implemented in a changing decision environment;
- The Fuzzy Inventory ORP Model provides the optimal solution in three scenarios and allows managers to select the most appropriate solution for their problem;
- The Fuzzy Inventory ORP Model helps managers in forecasting, due to the large numbers of the values for variables and coefficients considered through fuzzy numbers.

Weaknesses:

- The Fuzzy Inventory ORP Model deals only with positive fuzzy numbers.
- The Fuzzy Inventory ORP Model has to be defined considering the type of the objective function. In this paper, the ORP Model has a minimization objective function. In a maximization problem, the variables of the ORP model have to be transformed in such a way that the maximum values of the variables are related to the optimistic scenario and the minimum values of the variables are related to the pessimistic scenario. In real decision problems, there can be some difficulties on modelling the variables, especially when there are variables that have different rules for modelling.
- The model does not consider the fact that can be different ways to define the three scenario. In this model, the optimistic scenario is defined through minimum values for both variables and coefficients, and the pessimistic scenario is defined through maximum values for both variables and coefficients. This means that for a company it is optimistic to produce and hold in stock minimum quantities with minimum costs, because the objective function would achieve the minimum value of total costs. In reality, this is not always true. A company could consider more efficient and optimistic if it can produce or hold in stock large quantities of items with minimum costs. Therefore, the optimistic scenario, would have a different definition: ${q_i}^o$, ${s_i}^o$ and ${u_i}^o$ would be the maximum values of the fuzzy $\tilde{q}_i$, $\tilde{s}_i$ and $\tilde{u}_i$ and ${c_i}^o$, ${h_i}^o$ and ${k_i}^o$ would be the minimum values of the fuzzy costs: $\tilde{c}_i$, $\tilde{h}_i$ and $\tilde{k}_i$.

The ORP Model could be developed in order to consider different combination in defining the variables and scenarios. Also, it could be applied to the maximization problems by transforming it in PRO Model. This directions would be considered in subsequent research.

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